HARMONIC MAPPINGS BETWEEN SURFACES

Some local and global properties

by

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LIST OF CORRECTIONS TO J.C.WOOD'S THESIS

page

3 6 lines from bottom: For "$-\infty<\alpha<\beta<\infty$" read "$-\infty\leq\alpha<\beta\leq\infty$".

5 Line after (1.1J): For "volume n-form" read "volume m-form".

9 Lemma 1.2.4, first line: For "$k:D\to P$" read "$k:D\to\mathbb{R}$".

10 Theorem 1.2.6, third line: Before "$f:M\to N$" insert "a harmonic map".

12 Proof of 1.2.9, lines 3 & 4 Summation convention not to be applied here.

29 Formula (1.4B) Insert "$\leqslant$".

48 Right-hand diagram: For "D'","C'" read "C'","D'" respectively.

55 Corollary 2.2.2, line 2: For "$q\in U'$" read "$q\in U'\setminus p$".

59 Proof, line 3: Delete "2"'s in both formulae.

59 Proof, line 2: For "dJ(p)" read "dJ".

61 Line 12: For "or z=0" read "z'=0".

62 Second paragraph, first line: For "df(p)=0" read "df(p)$\neq$0",
second line: For "df(p)$\neq$0" read "df(p)=0".

70 First proof, first line: For "case (c) or case (d)" read "case 2 or case 3"

89 Formula: For "d" read "$|d|$".

91 Theorem 3.4.3, second line: For "f is monotone" read "f is monotone onto".

92 Remark 3.4.5, third line: For "degree 1 or 0" read "degree ±1 or 0".

96 Proof of 3.5.4: For "diffeomorphic" read "diffeomorphic via the exponential map".

102 First line: For "tangent to Σ" read "along Σ".

104 Line 2: Before "real analytic surfaces" insert "connected".

111 Proposition 4.2.1, first line: After "let M,N be compact analytic surfaces" insert ",N oriented,".

115 Proof, fourth line: For "$\chi(f(M))=0$" read "$\chi(f(M))\leqslant 0$", Delete "and we have equality in (4.2D)".

118 Paragraph 2, line 2: Replace "C^∞" by "C^1".

119 Top right-hand diagram: For "σC" read "τD", for "σD" read "τC".

123 Fifth line: For "image of C" read "interior of the image of C".

126 Line 2: After "holomorphic 1-forms" add "on M".

131 Line above (5.2B): For "deRham" read "Hodge".

131 5 lines from bottom: For "$f:M\to S^1$" read "$u:M\to S^1$". Last line: For "f" read "u".

132 Line 2: For "f" read "u".

140 First line: After "and" add "at most".

151 This page is continued on page 156.

152 Problem 5.3.8 (2): Add footnote "At corners, df must take normals to both sides to normals to E".

159 Lines 13,14: For "M" read "M_2".

ERRATA

Page

50 Replace this page by "Replacement p.50"

52 Remove this page

62 Last line: after "by definition" add "(*)". Add footnote: "(*) It can be seen from (2.2L) below that we can choose the neighbourhood U such that $f(U\setminus p)\subset N\setminus f(p)$."

63 Line after (2.2L): Replace "Now, by...Jordan regions" by "Now, by (2.1.20), p, f(p) are contained in relatively compact regions $U^*\subset M$, $V^*\subset N$ with $f(U^*\setminus p)\subset V^*\setminus f(p)$, U^* homeomorphic to an open disc and V^* a Jordan region"

68 Add further paragraph: "As specific examples of the types of good singular point in case (2) of theorem (2.2.11) we offer: (a) $u=x^2-y^2$, $v=y$, here the y-axis is a fold line; $u=xy$, $v=y$, here the x-axis is a collapse line; (b) $u = xy - \mathrm{re}(z^{k+1})$, $v=y$, $(2\leq k<\infty)$, here (0,0) is a good singular point of order k. A good singular point of order ∞ which is not a collapse point cannot occur for a harmonic map $\mathbb{R}^2\to\mathbb{R}^2$ by (2.1.17) (since such a map is C^ω); it is an important open question whether such points can exist for a smooth harmonic map $f:M\to N$ between Riemann surfaces."

69 Corollary (2.4.1): Replace by: "If p is a singular point for the mapping f, $f|_V$ is not injective on any neighbourhood V of p." Insert at beginning of proof: "Let V be any neighbourhood of p. Consider the mapping $f|:V\to N$. We shall show that there exists a neighbourhood U of p, $U\subset V$, such that $f|_U$ is not injective. It then follows that $f|_V$ is not injective."

81 Second line: Replace "if p has a ... local homeomorphism" by "if p has a neighbourhood U such that $f(U\setminus p)\subset N\setminus f(p)$ and $f|:U\setminus p\to N\setminus f(p)$ is a local homeomorphism"

81 Proposition (3.2.2) statement (A): Delete second line: "$\Leftrightarrow$ its branch set is empty or consists of isolated points". Proof (A): Replace proof by: "By (2.1.22), (2.1.24), f is ramified at p $\Leftrightarrow$ p is a branch point or f is a local homeomorphism at p. Therefore f is a ramified covering if and only if its branch set is empty or consists of branch points."

92-93 Delete the whole of section (3.4B)

99 Penultimate line: After "(x,y)" add "(*)". Add footnote: "(*) For convenience we have given this proof for dim(M)=2; the proof, however, extends to any dimension."

125 Definition (5.1.3), third line: Replace "if ψ=df..." by "if $\psi= (a_n z^n + a_{n+1} z^{n+1}+\dots)dz$, $a_n\neq 0$, is a meromorphic differential expressed in local complex coordinate z centred on p, then if $n>0$ (resp. <0), ψ is said to have a zero (resp. pole) of multiplicity n (resp. −n) at p."

K Types of singularities - branch points

Let $f:M\to N$ be a smooth map between connected smooth surfaces, and let $p\in M$.

Definition 2.1.19

We shall say that f is ramified at p if there exists a neighbourhood U of p such that $f(U\setminus p)\subset N\setminus f(p)$ and $f| : U\setminus p\to N\setminus f(p)$ is a local homeomorphism. (Thus for each $p'\in U\setminus p$, there exists a neighbourhood U' of p', $U'\subset U\setminus p$, and a neighbourhood V' of $f(p')$, $V'\subset N\setminus f(p)$, such that f maps U' homeomorphically onto V'.)

Note, as a special case, that f is ramified at p if it is a local homeomorphism at p.

By a Jordan region [A-S §1.20] we shall mean an open subset Δ of a surface whose closure can be mapped topologically onto a closed disc in such a way that Δ corresponds to the open disc.

Lemma 2.1.20

Let $f:M\to N$ be ramified at p. Then p, f(p) are contained in relatively compact regions $U^*\subset M$, $V^*\subset N$ with $f(U^*\setminus p)\subset V^*\setminus f(p)$ such that:

(1) U^* is homeomorphic to a disc, V^* is a Jordan region,

(2) the fundamental group of $V^*\setminus f(p)$ is infinite cyclic,

(3) the fundamental group of $U^*\setminus p$ is infinite cyclic,

(4) $f_*: \pi_1(U^*\setminus p)\to\pi_1(V^*\setminus f(p))$ is monomorphic.

Proof

Following Ahlfors & Sario [A-S §1.20] if U is a connected locally compact Hausdorff topological space and N is a C^0 surface, a continuous mapping $f\ U\to N$ is said to define U as a (ramified) covering surface of N if every $p'\in U$ has a neighbourhood $U'\subset U$ such that $f(U'\setminus p')\subset N\setminus f(p')$ and $f| : U'\setminus p'\to N\setminus f(p')$ is a local homeomorhism (*). Now let $f:M\to N$ be a smooth map between connected smooth surfaces, then it is clear that if f is ramified at p (in our sense) then there exists a neighbourhood U of p such that $f:U\to N$ defines U as an (Ahlfors & Sario) ...

(*) In Ahlfors & Sario terminology, $f:U'\setminus p'\to N\setminus f(p')$ is a "smooth covering."

ABSTRACT

We develop some local and global properties of a harmonic map $f:M\to N$ between surfaces. Our first main result is a local description of the possible singularities of such a harmonic map - we find there are four types: degeneracy, general fold, meeting point of general folds, branch point. As a corollary we have a result of Lewy and Heinz [Le1], [He1]. We show that the singularities of a harmonic map in higher dimensions can be qualitatively much nastier. We prove that there exist harmonic maps between compact surfaces exhibiting general folds.

Our second main result is an inequality arising from the Gauss-Bonnet formula relating the total curvature of the image of a harmonic map to its Euler Characteristic. We derive some corollaries of this inequality and compare with results obtained by convex function methods of Gordon [Go]. We also use the Gauss-Bonnet formula to show that if the codomain N has negative curvature, certain types of unnecessary or redundant folding cannot occur.

Other results include a characterisation of harmonic ramified coverings, an upper bound on the number of zeros of the derivative of a harmonic map, upper bounds on the number of singularities of different types for a harmonic map into the flat torus, a study of holomorphic maps into the sphere - such maps are harmonic [E-S], and some reflection principles for harmonic maps.

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CHAPTER 2. SINGULARITIES OF HARMONIC MAPS BETWEEN SURFACES

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0. INTRODUCTION AND STATEMENT OF MAIN RESULTS

A Local Results

(1) Let M,N be smooth connected orientable Riemannian manifolds without boundary whose smooth metrics are denoted by g,h respectively. We deal with harmonic maps $f:M\to N$ [E-S].

(2) If $\dim M = 2$, then f remains harmonic if g is replaced by any conformally equivalent metric (1.3.7), thus we can replace the metric on M by the induced complex structure.

(3) If $\dim M = \dim N = 2$, we can regard M and N as Riemann surfaces. For each choice of hermitian metric h on N, we have a notion of harmonic map independent of any choice of metric for M. Every holomorphic or anti-holomorphic map is harmonic with respect to any choice of hermitian metrics on M, N [E-S].

FOR THE REST OF THIS INTRODUCTION, M,N WILL BE CONNECTED RIEMANN SURFACES.

(4) If h is a C^∞ metric, then any harmonic map $f:M\to N$ is C^∞ [E-S] and our first main result is a local description of the possible singularities of f. Let Σ, Σ_2 denote the set of singular points of f, the subset of M on which df is zero, respectively.

Theorem 2.2.11

Let p be a singular point of f. Then there exists a neighbourhood U of p such that one of the following holds:

(1) $\Sigma\cap U = U$. Then U can be chosen so that either (a) f is constant on U, then $\Sigma_2\cap U = U$, or (b) $f(U)$ is a geodesic arc; arc length along the geodesic is a harmonic function on the neighbourhood U; U may further be chosen so that $\Sigma_2\cap U = \{p\}$ or $\emptyset$.

(2) $\Sigma\cap U$ is a general fold [W]. Then $\Sigma_2\cap U = \emptyset$. U can further be chosen such that either (a) $\Sigma\cap U$ is a fold line or a collapse line, or (b) p is a good singular point of order r $(2\leq r\leq\infty)$ (definitions in §2.1). If $\Sigma\cap U$ is a fold line, then if $f(\Sigma\cap U)$ has non-zero geodesic curvature at

$f(p)$, U may further be chosen such that f maps U to the convex side of the image $f(\Sigma \cap U)$ of the fold line.

(3) $\Sigma \cap U = \{p\} \cup$ even number of disjoint general folds with p an endpoint of each. Then p is a C^1 meeting point of these general folds (definition in §2.1J), and the general folds are arranged at equal angles around p. $\Sigma_2 \cap U = \{p\}$ or $\emptyset$.
Such points are isolated.

(4) $\Sigma \cap U = \{p\}$. Then
$\Sigma_2 \cap U = \{p\}$ and p is a branch point. Such points are isolated.

Method of Proof

In special coordinate systems on M,N we show that the harmonic map f can be expressed as harmonic polynomial + higher degree terms using results of Hartman & Wintner [H-W1]. We then examine the different cases and use results of Kuiper [Kp] on C^1-sufficiency of functions near isolated critical points and Sampson's maximum principle for harmonic maps [Sa].

Remarks

(1) Corollaries include results of Lewy and Heinz [Le1],[He1].

(2) The singularities of harmonic maps in higher dimensions can be qualitatively much worse as shown by examples (see §2.5B).

B Global Results

(1) If int $\Sigma_2 \neq \emptyset$, then f maps the whole of M to a point [Sa].

(2) If int $\Sigma \neq \emptyset$ and f is not constant, f maps M to a geodesic arc γ [Sa]; Σ_2 consists of the isolated critical points of f considered as a mapping $M \to \gamma$. Further if M is compact and of Euler characteristic $\chi(M)$, f maps M onto a closed geodesic, and card $\Sigma_2 \leqslant -\chi(M)$.

Combining (1) and (2) we see that if p is a singular point of the type (1) in theorem (2.2.11) above, then either f is constant on the whole of

M or f maps the whole of M to a geodesic arc.

(3) From (1) and (2) above we see that if rank $df(p) \leq r$ ($r \in \{0,1,2\}$) at all points p of an open subset of M, then rank $df(p) \leq r$ at all points of M. (3.1.9).

(4) If $\Sigma_1 = \emptyset$, then f is a ramified covering whose singularities consist entirely of branch points (see §3.2).

(5) If M,N are compact with genus(M) $\neq$ genus(N), any harmonic map $f:M \to N$ of degree ± 1 must have general folds.

(6) By consideration of a certain quadratic differential on M we show that for any harmonic map $f:M \to N$ with M compact and of Euler characteristic $\chi(M)$, if f is not holomorphic or anti-holomorphic, card $\Sigma_2 \leq -\chi(M)$. (1.3.14). Setting M = the Riemann sphere S shows that all non-constant harmonic maps $f:S \to N$ must be holomorhic or anti-holomorphic.(1.3.15). Setting M = any torus T (=compact Riemann surface of genus 1) shows that all harmonic maps $f:T \to N$ which are not holomorphic or anti-holomorphic have non-vanishing derivative.(1.3.16).

(7) If M is compact, f cannot have image in a convex supporting domain [Go]. We give some corollaries of this result.

C The Gauss Bonnet Inequality

Our second main result is

Theorem 4.2.3

Let M,N be compact Riemann surfaces, N with chosen real-analytic hermitian metric. Let $f:M \to N$ be a non-constant harmonic map. Then the total curvature $K(f(M))$ of the image of f is related to its Euler characteristic $\chi(f(M))$ by the inequality:

$$K(f(M)) \geq 2\pi\chi(f(M))$$

Method of Proof

We describe $f(M)$ as a semi-analytic set which allows us to apply the

Gauss-Bonnet formula to $f(M)$ as a subset of N. We then use Sampson's maximum principle[Sa] to obtain the inequality.

Corollaries

(1) $f(M)$ cannot have Euler characteristic $\geqslant 1$ (in particular cannot be contractible) unless its total curvature $\geqslant 2\pi$ or f is constant. (4.3.1).

(2) If N has non-positive curvature (e.g. $N=S^1 \times S^1$), $f(M)$ cannot have Euler characteristic $\geqslant 1$ (in particular cannot be contractible) unless f is constant. (4.3.2).

(3) Suppose N has strictly negative curvature on a dense subset of N, then, if $\chi(f(M)) \geqslant 0$ then f must either be constant or map M to a closed geodesic. In particular, $f(M)$ cannot be a tubular neighbourhood about a closed 1-submanifold of N. (4.3.4).

(4) If $N=S^2$ and $\chi(f(M)) \geqslant 1$ (e.g. $f(M)$ is contractible), $f(M)$ must cover half or more of the surface area of S^2 unless f is constant.

D Redundant Folds Theory

If M and N are compact connected Riemann surfaces, we may regard them as diffeomorphs of a sphere with attatched handles. A handle is a C^1 diffeomorph of $S^1 \times I$. Suppose that N has stricly negative curvature on a dense subset.

(1) Let $f: M \to N$ be a map that maps the handle H of M to the handle K of N in such a way that the end $S^1 \times \{0\} \subset H$ maps onto the end $S^1 \times \{0\} \subset K$ and the end $S^1 \times \{1\} \subset H$ maps onto the end $S^1 \times \{1\} \subset K$. Suppose the only singularities of f are a pair of disjoint closed fold lines "around" H (see (4.4.2) for exact hypotheses). Then f cannot be harmonic.

(2) Let $f: M \to N$ be a map that maps the handle H of M to the handle K of N in such a way that both ends $S^1 \times \{0\}$, $S^1 \times \{1\} \subset H$ map onto one end $S^1 \times \{0\} \subset K$. Suppose the only singularity of f is a closed fold line "around" H (see (4.4.4) for exact hypotheses). Then if f is harmonic, $\mathrm{int}(f(H))$ can contain no closed geodesic.

These results are obtained by applying the Gauss-Bonnet formula and Sampson's maximum principle in much the same way as in the proof of the Gauss-Bonnet inequality discussed above.

E Other Two-dimensional Results and Methods

Let M,N be connected Riemann surfaces as usual.

(1) Let P be a smooth Riemannian manifold of arbitrary dimension. Then the composition of a holomorphic or anti-holomorphic map $M \to N$ with a harmonic map $N \to P$ is harmonic. (1.3.9).

(2) If N = the two-sphere, S^2, we show how to construct holomorphic maps into S^2 using the Riemann-Roch theorem. Such maps are harmonic as mentioned before. (See § 5.1.)

(3) If N = the flat torus $S^1 \times S^1$, harmonic maps $M \to S^1 \times S^1$ may be obtained by integrating a pair of harmonic differentials on M. We obtain upper bounds on the number of general folds, branch points etc. by using these harmonic differentials. (See § 5.2.)

(4) We give some principles of reflection and doubling of harmonic maps. Some of these apply to harmonic maps between higher dimensional manifolds. (See § 5.3.)

(5) If M,N are smooth connected not necessarily orientable Riemannian manifolds of dimension 2, many of the above results concerning a harmonic map between connected Riemann surfaces can be generalised to the case of a harmonic map $f: M \to N$ by considering the lift of f to orientable double covers of M and N. (See for example (1.3.17).)

TERMINOLOGY & LIST OF SYMBOLS

Neighbourhood = open connected neighbourhood.

Domain = open connected set, Region = closure of a domain.

$\mathbb{R}$ = real numbers , $\mathbb{C}$ = complex numbers, $\mathbb{Z}$ = integers.

I = unit interval $[0,1]$ of $\mathbb{R}$. S^1 = Euclidean 1-sphere.

S^2 = Euclidean 2-sphere, S = Riemann sphere.

$\overline{}$ denotes closure of a subset of a topological space or complex conjugate of a complex number.

$\delta_{ij}, \delta^{ij}, \delta^i_j$ denote the Kronecker delta symbol equal to: 0 if $i \neq j$, 1 if $i=j$.

A C^r curve on a manifold M (of class $C^s, s \geq r$) is a C^r mapping $\tilde{\gamma}: (\alpha, \beta) \to M$ where $-\infty \leq \alpha < \beta \leq \infty$. We shall also call this a parametrised C^r arc in §1.1C.

If $f: M \to N$ is a C^1 mapping between manifolds M,N (of class $C^s, s \geq 1$) equipped with C^s local coordinates $(x^1, \ldots, x^m)$, $(u^1, \ldots, u^n)$, the derivative df locally has matrix form: $\begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^m} \\ \cdots & \cdots & \cdots \\ \frac{\partial f^n}{\partial x^1} & \cdots & \frac{\partial f^n}{\partial x^m} \end{bmatrix}$ If m=n, the Jacobian J is defined locally as the determinant of this matrix. Note that the equation J=0 is invariant of the local coordinates chosen.

Complex notation If (x,y) are local coordinates for a surface, we shall often use complex notation: $z=x+iy$, $|z| = \sqrt{x^2+y^2}$, re = real part, im = imaginary part, $\arg z = \tan^{-1}(y/x)$, regardless of whether (x,y) are complex coordinates (§1.3). We emphasise that this is merely a notational convenience which simplifies many formulae. For example we shall often use the fact that if $u = \mathrm{re}(a_k z^k)$ defines a real-valued function of (x,y), ($a_k \in \mathbb{C}$,) its partial derivatives are given by: $\frac{\partial u}{\partial x} = \mathrm{re}(k a_k z^{k-1})$, $\frac{\partial u}{\partial y} = -\mathrm{im}(k a_k z^{k-1})$.

1. BASIC PROPERTIES

1.1 Definition of a harmonic map

A The tension field

Let M,N denote connected smooth ($=C^\infty$) Riemannian manifolds without boundary of dimensions m,n, equipped with smooth metric tensors g,h and their Levi-Civita connections. (At times we shall specify that M,N and their metrics are real-analytic ($=C^\omega$).) Let $(x^1,\dots,x^m)$ denote smooth local coordinates for M in the neighbourhood of a point p, these are said to be centred on p if p has coordinates $(0,\dots,0)$; let $(u^1,\dots,u^n)$ denote smooth local coordinates for N. In these coordinates we may write the metrics on M,N as:

$$g = g_{ij}dx^i dx^j \qquad h = h_{\alpha\beta}du^\alpha du^\beta$$

Here and throughout this thesis we observe the summation convention and follow Hicks [Hi] when discussing differential geometry.

Let $f:M\to N$ be a C^2 map. Let TM,TN denote the tangent bundles of M,N; the map f may be used to form the pull-back bundle f^*TN, a bundle over M. Let $L^r(TM,f^*TN)$ denote the bundle over M of r - linear maps from TM to f^*TN. For a vector bundle E over M let $C^\infty(E)$ denote the vector space of smooth sections of E; then the derivative df of f may be considered as a member of $C^\infty(L^1(TM,f^*TN))$. Following Elliason [E1] or Eells-Sampson [E-S] we may form its covariant derivative $\nabla df \in C^\infty(L^2(TM,f^*TN))$ called the (second) fundamental form of f (*). Here ∇ is induced by the connections on M,N (see [E-S] p.121,[E1] p.174), in local coordinates, if the components of df are given by:

$$f_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}$$

then the components of ∇df are given by:

$$f_{;ij}^{\gamma} = \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - f_k^\gamma \Gamma_{ij}^{k} + f_i^\alpha f_j^\beta L_{\alpha\beta}^{\gamma} \tag{1.1A}$$

where Γ_{ij}^{k}, $L_{\alpha\beta}^{\gamma}$ denote the Christoffel symbols for M,N.

(*) Eells-Sampson use the notation $\tilde{\nabla}$ for covariant derivative ∇

Define now the trace of ∇df by the formula:

$$\mathrm{Tr}\, \nabla df(p) = \sum_{i=1}^{m} \nabla df(e_i, e_i)(p)$$

where $\{e_i\}$ is an orthonormal basis for the tangent space TM_p at $p \in M$; note $\mathrm{Tr}\nabla df \in C^\infty(f^* TN)$.

Definition 1.1.1

The tension field $\tau(f)$ of f is defined as the trace of the fundamental form of f, i.e:

(1.1B) $\quad \tau(f)(p) = \mathrm{Tr}\nabla df(p) = \sum_{i=1}^{m} \nabla df(e_i, e_i)(p).$ ∎

$\tau(f)$ can be considered as a mapping $M \to TN$ such that $\tau(f)(p) \in TN_{f(p)}$ - such mappings are called vector fields over f.

Proposition 1.1.2 [E-S]

In local coordinates the tension field is given by:

(1.1C) $\quad \tau(f)^\gamma = g^{ij}\left\{ \dfrac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - f_k^\gamma \Gamma_{ij}^{\ k} + f_i^\alpha f_j^\beta L_{\alpha\beta}^{\ \ \gamma} \right\}$ (†)

(1.1D) $\quad = \Delta f^\gamma + g^{ij} f_i^\alpha f_j^\beta L_{\alpha\beta}^{\ \ \gamma}$

where Δ is the Laplace-Beltrami operator for functions on M given by:

(1.1E) $\quad \Delta u = (\det g_{hk})^{-\frac{1}{2}} \dfrac{\partial}{\partial x^i}\left\{ g^{ij} \dfrac{\partial u}{\partial x^j} (\det g_{hk})^{\frac{1}{2}} \right\}$. ∎

Definition 1.1.3 [E-S] [Fu]

A C^2 map $f: M \to N$ is called harmonic if $\tau(f) = 0$. ∎

Thus f is harmonic if and only if, in any coordinates, it satisfies the harmonic equations:

(1.1F) $\quad g^{ij}\left\{ \dfrac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - f_k^\gamma \Gamma_{ij}^{\ k} + f_i^\alpha f_j^\beta L_{\alpha\beta}^{\ \ \gamma} \right\} = 0$

or eqivalently:

(1.1G) $\quad \Delta f^\gamma + g^{ij} f_i^\alpha f_j^\beta L_{\alpha\beta}^{\ \ \gamma} = 0$

Proposition 1.1.4 (see [E-S])

(1) Any C^2 map $f: M \to N$ between smooth Riemannian manifolds satisfying $\tau(f) = 0$ must be smooth.

(2) Any C^2 map $f: M \to N$ between real-analytic Riemannian manifolds satisfying $\tau(f) = 0$ must be real-analytic. ∎

Thus we shall only consider smooth or real-analytic mappings.

(†) g^{ij} denotes the inverse of g_{ij}, i.e. $g^{ij} g_{jk} = \delta^i_{\ k} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases}$

B $\tau(f)$ in normal coordinates

Let $p \in M$. We define normal coordinates centred on p as follows: let $\exp_p : TM_p \to M$ be the exponential map: $\exp_p(t)$ = endpoint of geodesic segment emanating from p and determined in length and direction by t; $\exp_p$ is a smooth map which maps a neighbourhood of $0 \in TM_p$ diffeomorphically onto a neighbourhood U of $p \in M$. Now choose a basis for TM_p. Define the coordinates of any point $q \in U$ as the components of $(\exp_p)^{-1}(q)$ with respect to the chosen basis. These are Normal coordinates centred at p (see also Hicks [Hi] p.131). If the basis for TM_p is orthonormal we shall say that the normal coordinates are orthonormal at p.

In terms of normal coordinates centred at p, the Christoffel symbols $\Gamma_{ij}^{\ k}(p) = 0$ (c.f. [E-S] p.117); if also the normal coordinates are orthonormal at p, then $g_{ij}(p) = \delta_{ij}$, hence we obtain the

Proposition 1.1.5 [E-S]

For a smooth map $f : M \to N$, in terms of p-centred normal coordinates for M which are orthonormal at p and f(p)-centred normal coordinates for N:

$$\tau(f)^{\gamma}(p) = \sum_{i=1}^{m} \frac{\partial^2 f^{\gamma}}{(\partial x^i)^2}(p) \,. \quad \blacksquare \tag{1.1H}$$

C Intepretation of the condition $\tau(f) = 0$

According to Chern & Goldberg [C-G] the fundamental form of f can be interpeted as follows:

By a parametrised arc of class C^r or parametrised C^r arc $(0 \leq r \leq \infty)$ we shall mean a C^r mapping $\tilde{\rho} : (\alpha, \beta) \to N$ from an open interval $(\alpha, \beta) \subset \mathbb{R}$ $-\infty \leq \alpha < \beta \leq \infty$. We often refer to the image $\rho = \tilde{\rho}(\alpha, \beta)$ as the arc and to $\tilde{\rho}$ as its parametrisation.

Let $\tilde{\rho} : (\alpha, \beta) \to \rho$, $s \mapsto \tilde{\rho}(s)$ be a parametrised C^2 arc; the acceleration vector of $\tilde{\rho}$ is the second covariant derivative of $\tilde{\rho}$ with respect to s, i.e:

$$\frac{D}{ds}\frac{d\tilde{\rho}}{ds}(s) \in TN_{\tilde{\rho}(s)}$$

In local coordinates:

(1.1I) $$\left\{\frac{D}{ds}\frac{d\tilde{\rho}}{ds}\right\}^{\gamma} = \frac{d^2\tilde{\rho}^{\gamma}}{ds^2} - \frac{d\tilde{\rho}^{\alpha}}{ds}\frac{d\tilde{\rho}^{\beta}}{ds}L_{\alpha\beta}^{\gamma}$$

If s measures arc length, the acceleration vector is just the geodesic curvature vector of ρ ([Hi] p.58).

Proposition 1.1.6 (intepretation of ∇df) [C-G]

Let e be a unit vector in TM_p and let γ be the geodesic tangent to e at p, parametrise γ by arc length s and let $s \mapsto \tilde{\gamma}(s)$ denote the parametrisation. Then $\nabla df(e,e)$, considered as a vector in TN_p, is the acceleration vector of the parametrised arc $f\circ\gamma : s \mapsto f(\tilde{\gamma}(s))$.

We shall call this acceleration vector simply the acceleration vector of $f(\gamma)$.

Proof

Take p-centred normal coordinates $(x^1,\ldots,x^n)$ orthonormal at p with the x^1-axis the geodesic γ tangent to e. Then from (1.1A) we calculate the components of $\nabla df(e,e)$ as:

$$f_{;11}^{\gamma} = \frac{\partial^2 f^{\gamma}}{(\partial x^1)^2} - f_k^{\gamma}\Gamma_{11}^{k} + f_1^{\alpha}f_1^{\beta}L_{\alpha\beta}^{\gamma}$$

But since the coordinates are normal, $\Gamma_{11}^{k}(p) = 0 \ \forall k$, therefore:

$$f_{;11}^{\gamma} = \frac{\partial^2 f^{\gamma}}{(\partial x^1)^2} - f_1^{\alpha}f_1^{\beta}L_{\alpha\beta}^{\gamma}$$

and since x^1 measures arc length along γ, comparing with (1.1I) shows that these are the components of the acceleration vector of $f\circ\tilde{\gamma}$. ∎

Proposition 1.1.7 (intepretation of $\tau(f)=0$)

Let $\{e_1,\ldots,e_m\}$ be an orthonormal basis in TM_p and let $\gamma_1,\ldots,\gamma_m$ be geodesics tangent to $e_1,\ldots,e_m$ at $p\in M$. Then f is harmonic if and only if the average of the acceleration vectors of the $f(\gamma_i)$ is zero.

Proof

$$\tau(f)(p) = \mathrm{Tr}\,\nabla df(p) = \sum_{i=1}^{m}\nabla df(e_i,e_i)(p)$$

= m times the average of the acceleration vectors of the $f(\gamma_i)$ ∎

Thus a harmonic map is one which maps geodesics to geodesics "on average".

A special example of a harmonic map is a map whose fundamental form ∇df is zero. Such a map is called totally geodesic. From the above intepretation of ∇df, we see that a map is totally geodesic if and only if it maps geodesics to geodesics linearly.

Notes 1.1.8

(1) If $N=\mathbb{R}$, the definition of harmonic accords with the usual definition of a harmonic function $f:N\to\mathbb{R}$ as one whose Laplacian Δf is zero.

(2) If $M=\mathbb{R}$, $\tau(f)(p)$ is just the geodesic curvature of the parametrised arc $f:\mathbb{R}\to N$, thus f is harmonic if and only if it defines a geodesic. Thus the harmonic maps problem is sometimes called "the generalised elastic band problem".

D Energy of a mapping

We now give an interpretation of a harmonic map as the extremal of a certain functional. For this interpretation assume that M is compact and oriented and that N is complete.

Definition 1.1.9 [E-S]

The energy(integral) of a C^2 mapping $f:M\to N$ is the non-negative number:

(1.1J) $$E(f) = \tfrac{1}{2}\int_M \|df\|^2 *1$$

Here $*1$ denotes the volume m-form on M canonically determined by the metric and orientation, and $\| \ \|$ denotes the norm on the bundle $L^1(TM,f^*TN)$ canonically determined by the metrics on M,N. Explicitly:

(1.1K) $$\|df\|^2(p) = \sum_{i=1}^{m}\|df(e_i)\|_N^2(p)$$

where $\{e_i\}$ is an orthonormal basis for TM_p and $\| \ \|_N$ denotes the norm on the tangent space $TN_{f(p)}$ defined by the metric on N. In local coordinates:

$$\|df(e_i)\|_N^2(p) = h_{\alpha\beta}f_i^{\alpha}f_i^{\beta} \qquad \forall i$$

and

$$\|df\|^2(p) = g^{ij}h_{\alpha\beta}f_i^{\alpha}f_j^{\beta} \quad \blacksquare$$

Proposition 1.1.10 [E-S] [Sm]

A C^2 map $f:M\to N$ is harmonic if and only if it is an extremal of the energy integral $E(f)$.

(For details see [E-S], for an extension to the case when M may not be compact see [C-G]. ∎

E Interpretation of $E(f)$

Let $\{e_1, \ldots, e_m\}$ be an orthogonal set of tangent vectors in TM_p, then the integrand (1.1K) of the energy integral can be intepreted as the sum of the squares of the stretches in length $\frac{\|df(e_i)\|_N}{\|e_i\|_M}$ caused by the mapping f. Thus minimising $E(f)$ can be thought of as minimising the overall stretching of the map f. (See also [E-S]p.114).

F Existence of harmonic maps

We now discuss the question of whether an arbitrary map can be deformed into a harmonic map.

Theorem 1.1.11 (Existence theorem) (Eells & Sampson, [E-S], Hartman [Ha])

If M is a compact connected smooth Riemannian manifold and

N is a complete connected smooth Riemannian manifold with non-positive sectional curvature, then every homotopy class of mappings $M \to N$ contains a harmonic one. ∎

Remark

As noted by Sampson [Sa] no orientability assumption on M is required.

Theorem 1.1.12 (Uniqueness Theorem) (Hartman [Ha] theorem I)

If $f: M \to N$ is harmonic, where M is a compact connected smooth Riemannian manifold and N is a complete connected smooth Riemannian manifold with strictly negative sectional curvature, then f is the only harmonic map in its homotopy class unless (1) f is constant, or (2) f maps into a geodesic arc γ, in which case all harmonic maps homotopic to f are obtained by a 'rotation' of f, I.e. by moving each point $f(p)$ a fixed oriented distance along γ, and conversely, every rotation of f gives a harmonic map in the homotopy class. ∎

Remarks

(1) Sampson ([Sa] theorem 4) slightly relaxes the curvature assumption on N.

(2) With M,N asin theorem (1.1.12), Harmonic maps give canonical representatives for each homotopy class of mappings $M \to N$. It is a fundamental unsolved question whether the existence theorem is true without a curvature assumption on N, e.g. if N is an n-sphere. Many examples of harmonic maps between spheres have been given by R.T Smith [Sm] and R. Wood [Wd]. We shall discuss harmonic maps of surfaces into the 2-sphere in §5.1.

1.2 Compositions

A The composition law

Let M,N,P be smooth Riemannian manifolds. In general, the composition of two harmonic maps $f:M\to N$, $k:N\to P$ is not harmonic (for example see [E-S] p.132) without further restrictions on one or other of the maps.

Proposition 1.2.1 (Composition law) [E-S]p.131

The fundamental form and tension field of the composite of two smooth maps $f:M\to N$, $k:N\to P$ are given in local coordinates by:

(1.2A) $\quad (k\circ f)_{;ij}{}^{a} = k_{\gamma}{}^{a} f_{;ij}{}^{\gamma} + k_{;\alpha\beta}{}^{a} f_i{}^{\alpha} f_j{}^{\beta}$

(1.2B) $\quad \tau(k\circ f)^{a} = k_{\gamma}{}^{a}\tau(f)^{\gamma} + g^{ij}k_{;\alpha\beta}{}^{a} f_i{}^{\alpha} f_j{}^{\beta}$

Proof

Direct computation. ∎

We now derive a coordinate-free expression for $\tau(k\circ f)$. Recall that $\tau(f)$ can be considered as a mapping $M\to TN$; we may regard the derivative dk of k as a linear bundle map $TN\to TP$ and its covariant derivative ∇dk as a linear bundle map: $TN\otimes TN\to TP$. Then (1.2B) can be written in the form (c.f. [Sm]p.9):

(1.2C) $\quad \tau(k\circ f) = dk\circ \tau(f) + \mathrm{Tr}\,\nabla dk(df,df)$

where $\nabla dk(df,df)$ is the composition:

$$TM\otimes TM \xrightarrow{df\otimes df} TN\otimes TN \xrightarrow{\nabla dk} TP$$

Thus if $p\in M$:

(1.2D) $\quad \mathrm{Tr}\,\nabla dk(df,df)(p) = \sum_{i=1}^{n}\nabla dk(df(e_i),df(e_i))(p)$

where $\{e_1,\dots,e_n\}$ is an orthonormal basis for TM_p.

Corollary 1.2.2 [E-S]

If f is harmonic and k is totally geodesic, then $k\circ f$ is harmonic.

Proof

In (1.2C), $\tau(f) = 0$, $\nabla dk = 0$, hence $\tau(k\circ f) = 0$. ∎

Thus new harmonic maps may be manufactured from old ones by composing with totally geodesic maps, in practise this is of limited use as there are few totally geodesic maps. As for compositions 'in the other direction' we shall see in §1.3 that a holomorphic map between surfaces followed by a harmonic map is harmonic. See Smith [Sm] for further results on compositions.

B Compositions with convex functions

Let N be a smooth Riemannian manifold. A smooth function $k:D\to\underline{\mathbb{R}}$ defined on a domain $D\subset N$ is said to be convex (resp. strictly convex) iff its fundamental form (often called the Hessian) ∇dk is positive semi-definite (resp. positive definite). From (1.1A) the Hessian of k has components:

$$k_{;\alpha\beta} = \frac{\partial^2 k}{\partial x^\alpha \partial x^\beta} - k_\gamma L_{\alpha\beta}{}^\gamma$$

Proposition 1.2.3

$k:D\to\underline{\mathbb{R}}$ is convex (resp. strictly convex) if and only if on each geodesic $\tilde{\sigma}:(\alpha,\beta)\to D\subset N$, $(-\infty\leq\alpha<\beta\leq\infty)$ parametrised by arc length s, $\frac{d^2}{ds^2}(k\circ\tilde{\sigma})\geq 0$ (resp. >0).

Proof

By (1.1.6), if e is a unit vector in TN_b $(b\in N)$ and $\tilde{\sigma}:(\alpha,\beta)\to D$ is the geodesic tangent to e at b, parametrised by arc length s, then $\nabla dk(e,e)$ is the acceleration vector of $k\circ\tilde{\sigma}$, i.e. $\nabla dk(e,e) = \frac{d^2}{ds^2}(k\circ\tilde{\sigma})$. Thus $\nabla dk(e,e)\geq 0$ (resp. >0) if and only if $\frac{d^2}{ds^2}(k\circ\tilde{\sigma})\geq 0$ (resp. >0). The result follows. ∎

Convex functions arise naturally as squared distances, especially on manifolds of negative curvature [B-O'N].

Now let M,N,P be connected smooth Riemannian manifolds.

Lemma 1.2.4

If $f:M\to N$ is harmonic, $k:D\to\underline{\mathbb{R}}$ is convex and $f(M)\subset D$, then $k\circ f$ is a subharmonic function on M i.e. $\Delta(k\circ f)\geq 0$ where Δ is the Laplace-Beltrami operator for functions on M given by (1.1E).

Proof

From the composition law:

$$\Delta(k\circ f) = \tau(k\circ f) = dk\circ\tau(f) + \mathrm{Tr}\,\nabla dk(df,df)$$

But $\tau(f) = 0$ and ∇dk is positive semi-definite, hence $\mathrm{Tr}\,\nabla dk(df,df)\geq 0$ and so $\Delta(k\circ f)\geq 0$. ∎

Lemma 1.2.5 [P-W]

If M is compact, and $K:M\to\underline{\mathbb{R}}$ is a smooth function on M,

$$\int_M \Delta K * 1 = 0 \,. \quad ∎$$

Theorem 1.2.6 (Gordon [Go])

Call a subset D of N convex supporting if every compact subset of D has an (open) neighbourhood in N which supports a strictly convex function.

Then, if M is compact, a harmonic map $f:M\to N$ cannot have image in a convex supporting subset unless f is constant.

Proof

If f is harmonic and k is convex then by (1.2.4) $k\circ f$ is subharmonic on M i.e. $\Delta(k\circ f)\geqslant 0$ throughout M. But by (1.2.5), $\int_M \Delta(k\circ f)*1 = 0$.

We conclude that $\Delta(k\circ f)\equiv 0$ on M. But by the composition law, c.f. (1,2.4):

$$\Delta(k\circ f) = \mathrm{Tr}\,\nabla dk(df,df)$$

Therefore $\mathrm{Tr}\,\nabla dk(df,df)\equiv 0$. But since k is strictly convex, ∇dk is positive definite. It follows that $df\equiv 0$ and therefore, f is constant.

We shall explore some corollaries of this in §3.5.

C Sampson's maximum principle [Sa]

Let M,N be a smooth connected Riemannian manifolds.

Lemma 1.2.7 [P-W p.61] (Maximum Principle for the Laplacian)

A non-constant subharmonic function F defined on an open subset of M can have no local maximum point. (Here a local maximum point is a point $p\in M$ such that $F(q)\leqslant F(p)$ $\forall q\in$ some neighbourhood of p.)

Let $S\subset N$ be a hypersurface i.e. a smooth submanifold of codimension 1; let $i:S\to N$ denote the inclusion map; using (1.1.6) we can intepret the fundamental form ∇di as follows: let $b\in S$ and let e be a unit vector tangent to S at b. Let σ be the geodesic in S tangent to e at b. Then by (1.1.6) and the immediately preceeding remark, $\nabla di(e,e)$ = geodesic curvature vector of the arc $i(\sigma)\subset N$. (It can be shown [Hi] that $\nabla di(e,e)$ is normal to S.) Now the tangents TS_b to the hypersurface S form a hyperplane dividing TN_b into two open half-spaces S_1,S_2. Suppose, as e varies over all unit vectors tangent to s at b, $\nabla di(e,e)$ is a vector ...

which always lies in the same half-space, say S_1, then we shall say that the fundamental form of $i:S\subset N$ is <u>Definite</u> at b. Choose a neighbourhood U of TS_b in TN_b such that the exponential map $\exp_b:TN_b\to N$ is a smooth diffeomorphism onto a neighbourhood V of b. Then the neighbourhood $\exp_b U$ of b is divided into two halves: $\exp_b(U\cap S_1)$ and $\exp_b(U\cap S_2)$. If $\nabla di(e,e)\in S_1 \;\forall e$, we say that $U\cap S_1$ is the <u>concave</u> side of S_1 and $U\cap S_2$ is the <u>convex</u> side of S (near b) (If the second fundamental form is not definite we cannot talk about a concave side or a convex side.)

<u>Remark</u>

If N is two-dimensional so that S is a 1-dimensional submanifold, then if e is a unit vector tangent to S at $b\in S$, $\nabla di(e,e)$ is, by (1.1.6), the geodesic curvature vector of S. Then the fundamental form of i, ∇di, is definite at b if and only if the geodesic curvature at b is non-zero. ∎

We now construct a convex function in a neighbourhood of b which is zero on the hypersurface S.

Firstly we construct coordinates $(u^1,\dots,u^n)$ centred on b as follows: take $(u^2,\dots,u^n)$ to be normal coordinates for the manifold S so that the axes are geodesics of S, and let the point $(u^1,\dots,u^n)$ be the point a directed distance u^1 along the unique geodesic of N which passes through the point $(u^2,\dots,u^n)$ and is perpendicular to S.

<u>Lemma 1.2.8</u>

$$L_{1\alpha}^{\;1}(b) = 0 \quad (\alpha=1,\dots,n)$$

<u>Proof</u>

From Hicks [Hi] at $b\in S$,

$$L_{1\alpha}^{\;1} = \tfrac{1}{2}h^{1r}\left\{\frac{\partial h_{r\alpha}}{\partial x^1}+\frac{\partial h_{r1}}{\partial x^\alpha}-\frac{\partial h_{1\alpha}}{\partial x^r}\right\}$$

But $h_{11}=h^{11}=1$, by definition of u^1, and $h_{1r}=h^{1r}=0$ $(r=2,\dots,n)$ since the lines $u^2=\dots=u^n=$constant are perpendicular to S. Therefore:

$$L_{1\alpha}^{\;1} = \tfrac{1}{2}h^{11}\left\{\frac{\partial h_{1\alpha}}{\partial x^1}+\frac{\partial h_{11}}{\partial x^\alpha}-\frac{\partial h_{1\alpha}}{\partial x^1}\right\} = 0 \quad (\alpha=1,\dots,n) \quad \text{at } b\in S. \;∎$$

Lemma 1.2.9

The (second) fundamental form of $i:S\subset N$ has components:

$$i_{;\alpha\beta}{}^{1} = L_{\alpha\beta}{}^{1} \quad (\alpha,\beta=2,\dots,n) \qquad i_{;\alpha\beta}{}^{\gamma} = 0 \quad (\alpha,\beta,\gamma=2,\dots,n)$$

Proof

In the coordinates $(u^2,\dots,u^n)$ for S, $(u^1,\dots,u^n)$ for N, the map i has components: $i^1=0$, $i^2=u^2,\dots,i^n=u^n$; thus from (1.1A):

$$i_{;\alpha\beta}{}^{1} = 0 - 0 + i_{\alpha}^{\alpha'} i_{\beta}^{\beta'} L_{\alpha'\beta'}{}^{1} = L_{\alpha\beta}{}^{1} \quad (\alpha,\beta=2,\dots,n)$$

$$i_{;\alpha\beta}{}^{\gamma} = 0 - i_{\gamma'}^{\gamma} L_{\alpha\beta}{}^{\gamma'} + i_{\alpha}^{\alpha'} i_{\beta}^{\beta'} L_{\alpha'\beta'}{}^{\gamma} \quad (\alpha,\beta,\gamma=2,\dots,n) \ \blacksquare$$

Lemma 1.2.10

The function $(u^1,\dots,u^n)\mapsto u^1$, defined on a neighbourhood of $b\in N$, has (second) fundamental form (expressed as a partitioned matrix) at b:

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & -L_{\alpha\beta}{}^{1} \end{array}\right]$$

where the matrix $(-L_{\alpha\beta}{}^{1})_{\alpha,\beta=2,\dots,n}$ is positive (resp. negative) definite if S is convex (resp. concave) on the side $u^1>0$ near b.

Proof

From (1.1A), at b:

$$u_{;\alpha\beta}{}^{1} = -u_1{}^{1} L_{\alpha\beta}{}^{1} = -L_{\alpha\beta}{}^{1} \quad (\alpha,\beta=1,\dots,n)$$

By lemma (1.2.8) this has the form shown. By lemma (1.2.9) the matrix $(L_{\alpha\beta}{}^{1})_{\alpha,\beta=2,\dots,n} = (i_{;\alpha\beta}{}^{1})_{\alpha,\beta=2,\dots n}$ is the fundamental form of i.

If S is convex (resp. concave) on the side $u^1>0$, the matrix $(i_{;\alpha\beta}{}^{1})$ is negative (resp. positive) definite. Hence $(-L_{\alpha\beta}{}^{1})_{\alpha,\beta=2,\dots n}$ is positive (resp. negative) definite. $\blacksquare$

We now modify the function $(u^1,\dots,u^n)\mapsto u^1$ to give a strictly convex function.

Lemma 1.2.11

If S has definite (second) fundamental form, we can define a strictly convex function in a neighbourhood of $b\in S$ which is <0 on the concave side of S , >0 on the convex side of S and $=0$ on S.

Proof

With coordinates $(u^1,\dots,u^n)$ as above, we may assume with no loss of generality that S is convex on the side $u^1>0$. Now let $k(u^1)$ be a strictly increasing and strictly convex function of u^1 with $k(0)=0$, defined in a neighbourhood of $u^1=0$. By the composition law (1.2A), at $b\in S$:

$$(k\circ u^1)_{;\alpha\beta} = \frac{dk}{du^1}\, u_{;\alpha\beta}{}^1 + \frac{d^2k}{du^2}\, u_\alpha{}^1 u_\beta{}^1$$

Hence:

$$(k\circ u^1)_{;11} = \frac{dk}{du^1}\, u_{;11}{}^1 + \frac{d^2k}{du^2}\, u_1{}^1 u_1{}^1 = \frac{d^2k}{du^2}$$

since by (1.2.10), $u_{;11}{}^1 = 0$ and $u_1{}^1 = 1$.

Also,

$$(k\circ u^1)_{;\alpha\beta} = \frac{dk}{du^1}\, u_{;\alpha\beta}{}^1 \quad ,(\alpha,\beta) \neq (1,1)$$

since $u_\alpha{}^1 = 0$ $(\alpha\neq 1)$.

Thus the hessian of $k\circ u^1$ at $b\in S$ has the form:

$$\left[\begin{array}{c|c} \dfrac{d^2k}{(du^1)^2} & 0 \\ \hline 0 & \dfrac{dk}{du^1}\, u_{;\alpha\beta}{}^1 \end{array}\right]$$

Now, since k is a strictly increasing and strictly convex function of u^1, $\frac{d^2k}{(du^1)^2} > 0$, and $\frac{dk}{du^1} > 0$, and by (1.2.10), $(u_{;\alpha\beta}{}^1)_{\alpha,\beta=2,\dots,n}$ is positive definite. We thus see that the Hessian of $k\circ u^1$ is positive definite at $b\in S$ thus $k\circ u^1$ is strictly convex near b. Also, since k is a strictly increasing function of u^1 with $k(0) = 0$, $k <,=,> 0$ according as $u^1 <,=,> 0$ i.e. on the concave side of S, on S, on the convex side of S respectively. ∎

Theorem 1.2.12 (Maximum principle) (Sampson [Sa])

Let M,N be smooth connected Riemannian manifolds. Let $S\subset N$ be a hypersurface with definite fundamental form at $b=f(p)$. If $f:M\to N$ is harmonic, then no neighbourhood of p is mapped entirely on the concave side of S.

Proof

For construct a strictly convex function k in a neighbourhood of b as in lemma (1.2.11), then by lemma (1.2.4), $k\circ f$ is .

subharmonic on a neighbourhood of $p\in M$. Suppose a neighbourhood of p is mapped entirely on the concave side, then since $k<0$ on the concave side of S, and $k=0$ on S, $k \circ f$ would have a local maximum at p. By (1.2.7) this cannot happen unless $k \circ f$ is constant. Thus no neighbourhood of p is mapped entirely on the concave side of S.∎

Remarks 1.2.13

If N is two-dimensional, then,as remarked earlier,the hypersurface $S \subset N$ has definite fundamental form if and only if it has non-zero geodesic curvature.

1.3 Special properties in two dimensions

A Riemann Surfaces

Let M be a connected smooth oriented two-dimensional manifold equipped with a smooth Riemannian metric g. Two such metrics g, g' are said to be conformally equivalent if $g = \lambda g'$ for some smooth positive function $\lambda : M \to \mathbb{R}$. In smooth local coordinates (x^1, x^2) on M, g, g' are conformally equivalent iff there exists a smooth positive real-valued function λ on each coordinate neighbourhood such that $g_{ij} = \lambda g'_{ij}$, or equivalently, $g^{ij} = \lambda^{-1} g'^{ij}$. We define a conformal structure on M as a conformal equivalence class of smooth metrics g.

Thus any metric determines a unique conformal structure; we shall call a connected smooth oriented two-dimensional manifold equipped with a conformal structure a Riemann surface. Thus a Riemann surface M is a smooth oriented manifold together with a conformal equivalence class of smooth metrics. We call a smooth metric g on M hermitian if it lies in the prescribed conformal equivalence class.

Note

If M is a connected smooth unorientable two-dimension manifold, we may consider instead its orientable double cover.∎

Now let M be a Riemann surface; choose a hermitian metric g on M.

Definition 1.3.1

Let $p \in M$. the angle between two tangent vectors $t, t' \in TM_p$ is the number θ given by:

$$(1.3A) \qquad \theta = \cos^{-1}\left\{\frac{\langle t, t'\rangle}{\|t\|\|t'\|}\right\} \qquad (0 \leq \theta \leq \pi)$$

where $\langle\ ,\ \rangle$ and $\|\ \|$ denote the inner product and norm, respectively, on TM_p determined by the metric g, with $\|t\| = \sqrt{\langle t, t\rangle}$.

In local coordinates (x^1, x^2), if $t = t^i \frac{\partial}{\partial x^i}$, $t' = t'^i \frac{\partial}{\partial x^i}$, then (1.3A) can be written:

$$(1.3B) \qquad \theta = \cos^{-1}\left\{\frac{g_{ij} t^i t'^j}{\sqrt{\{g_{ij} t^i t^j\}}\sqrt{\{g_{ij} t'^i t'^j\}}}\right\}$$

Proposition 1.3.2

The angle θ is independent of the particular choice of hermitian metric on M.

Proof

If g' is another hermitian metric, then $g' = \lambda g$. Using subscripts to distinguish between the inner products determined by g, g', we have:

$$\theta_{g'} = \cos^{-1}\left\{\frac{\langle t,t'\rangle_{g'}}{\|t\|_{g'}\|t'\|_{g'}}\right\} = \cos^{-1}\left\{\frac{\lambda\langle t,t'\rangle_g}{\sqrt{\lambda}\|t\|_g\sqrt{\lambda}\|t'\|_g}\right\}$$
$$= \cos^{-1}\left\{\frac{\langle t,t'\rangle_g}{\|t\|_g\|t'\|_g}\right\} = \theta_g$$

and thus we see that θ is independent of the particular choice of hermitian metric on M. ∎

Thus on a Riemann surface the concept of angle is well-defined. We now study special coordinate systems on M:

Definition 1.3.3

Let (x^1,x^2) be smooth local coordinates defined in a neighbourhood of a point $p \in M$. We say (x^1,x^2) are isothermal at p if for any hermitian metric g, $g_{ij}(p) = \text{const}.\delta_{ij}$ $(i,j=1,2)$, or equivalently, $g_{11}(p) = g_{22}(p)$ and $g_{12}(p) = 0$, or equivalently, $g^{11}(p) = g^{22}(p)$ and $g^{12}(p) = 0$. ∎

Clearly this definition is independent of the choice of hermitian metric.

We say the local coordinates (x^1,x^2) are isothermal if they are isothermal at every point. Thus in isothermal coordinates a hermitian metric g assumes the form:

$$g = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 = \rho((dx^1)^2 + (dx^2)^2)$$

where ρ is a smooth positive function defined on the coordinate neighbourhood. The existence of isothermal coordinates is well-known [Ch].

Lemma 1.3.4

The angle between t,t' in isothermal coordinates (x^1,x^2) is

$$\theta = \cos^{-1}\left\{\frac{t^1t'^1 + t^2t'^2}{\sqrt{\{(t^1)^2+(t^2)^2\}\{(t'^1)^2+(t'^2)^2\}}}\right\}$$

Proof

From (1.3B), putting $g_{ij} = \text{const}.\delta_{ij}$. ∎

This lemma will be needed later.

For ease of notation we shall now denote coordinates by (x,y) in place of (x^1,x^2).

Proposition 1.3.5 [Sp p.21]

Let (x,y), (x',y') be isothermal coordinates defined in a neighbourhood of some point $p\in M$. Then the change of coordinates $(x,y)\mapsto(x',y')$ is C^{ω}; indeed writing $z=x+iy$, $z'=x'+iy'$, the change of coordinates $z\mapsto z'$ is holomorphic or antiholomorphic according as the change of coordinates is orientation preserving or reversing. ∎

Isothermal coordinates (x,y) compatible with the orientation of M are called complex coordinates. Complex coordinates may be obtained from isothermal coordinates by replacing (x,y) by $(-x,y)$ where necessary. The transition functions between overlapping charts are holomorphic, thus we have introduced a complex analytic structure onto M. (*)

We shall often use complex notation $z=x+iy$, $\bar{z}=x-iy$.

A smooth mapping $f:M\to N$ is holomorphic (resp. antiholomorphic) iff it is holomorphic (resp. antiholomorphic) in the local complex coordinates on M, N. Writing these coordinates as (x,y), (u,v) and using complex notation $z=x+iy$, $w=u+iv$, a smooth map f is holomorphic (resp. antiholomorphic) iff $\frac{\partial w}{\partial \bar{z}}=0$ (resp. $\frac{\partial w}{\partial z}=0$). Here: $\frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left\{\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right\}$, $\frac{\partial}{\partial z}=\frac{1}{2}\left\{\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right\}$.

Holomorphic and antiholomorphic mappings f have the property of conformality which may be stated as follows: if $df(p)\neq 0$ (at $p\in M$) and $t_1,t_2\in TM_p$, $t_1,t_2\neq 0$, then (1) the angle between t_1 and t_2, (2) the ratio of the norms of t_1,t_2 are preserved under the mapping. I.e. if g,h are any hermitian metrics on M,N respectively:

(*) A Riemann surface is often defined to be a (connected) manifold with a complex analytic structure [A-S], [Sp]. We have shown that a Riemann surface in our sense is certainly a Riemann surface in this sense. Note however that our definition is more restrictive, our Riemann surfaces being required to possess a globally defined hermitian metric. However if M is a paracompact complex-analytic manifold, partitions of unity may be used to establish the existence of globally defined hermitian metrics.

$$(1.3C) \qquad \cos^{-1}\frac{\langle t_1,t_2\rangle_M}{\|t_1\|_M\|t_2\|_M} = \cos^{-1}\frac{\langle df(p)(t_1),df(p)(t_2)\rangle_N}{\|df(p)(t_1)\|_N\|df(p)(t_2)\|_N}$$

and

$$(1.3D) \qquad \frac{\|t_1\|_M}{\|t_2\|_M} = \frac{\|df(p)(t_1)\|_N}{\|df(p)(t_2)\|_N}$$

B Harmonic maps from a Riemann surface

Let P be a smooth Riemannian manifold of any dimension, and let M be a Riemann surface equipped with local isothermal coordinates (x,y) and a chosen hermitian metric g.

Lemma 1.3.6

If we write $g=\rho(dx^2+dy^2)$ $\rho>0$, then the tension field of a smooth map $f:M\to P$ is given by:

$$(1.3E) \qquad \tau(f)^\gamma = \frac{1}{\rho}\left\{\frac{\partial^2 f^\gamma}{\partial x^2}+\frac{\partial^2 f^\gamma}{\partial y^2} + f_i^\alpha f_j^\beta L_{\alpha\beta}{}^\gamma\right\}$$

Proof

Directly from (1.1D) using the well-known expression:

$$\Delta f^\gamma = \frac{1}{\rho}\left\{\frac{\partial^2 f^\gamma}{\partial x^2}+\frac{\partial^2 f^\gamma}{\partial y^2}\right\} \quad \blacksquare$$

Proposition 1.3.7

Harmonicity of $f:M\to P$ is independent of the choice of hermitian metric g on M.

Proof

If $g=\rho(dx^2+dy^2)$ and $g'=\rho'(dx^2+dy^2)$ aretwo hermitian metrics, and $\tau(f)$, $\tau'(f)$ are the corresponding tension fields for f, (1.3E) shows that $\tau'(f)=\frac{\rho}{\rho'}\tau(f)$. Since $\rho>0$, $\rho'>0$, $\tau(f)=0$ if and only if $\tau'(f)=0$. $\blacksquare$

Remarks

(1) Eells & Sampson show that the energy integral $E(f)$ is also independent of the choice of hermitian metric.

(2) Thus for a smooth map $f:M\to P$ from a Riemann surface M we may talk about f being harmonic without needing to specify any particular hermitian metric on M.

Corollary 1.3.8

Any C^2 mapping $f:M\to P$ from a Riemann surface M to a C^ω Riemannian manifold P which satisfies $\tau(f)=0$ must be C^ω with respect to the complex structure on M.

Proof

Let $p\in M$, choose local isothermal coordinates (x,y) in a neighbourhood U of p and choose a hermitian metric on U of the form $\lambda(dx^2+dy^2)$ where $\lambda:U\to\mathbb{R}^{>0}$ is C^ω. Then the result follows by applying (1.1.4) to each such coordinate neighbourhood U. ∎

Thus for harmonic mappings from surfaces , we only need to know that the target manifold is C^ω to conclude f is C^ω (w.r.t. the complex structure on the surface.)

C A composition law

Let M,N be Riemann surfaces equipped with smooth hermitian metrics g,h. Eells & Sampson prove that a holomorphic or antiholomorphic map $f:M\to N$ is harmonic with respect to any choice of hermitian metrics g,h [E-S p.118]; further Lichnerowitz [Li] shows that such a map gives an absolute minimum of the energy integral $E(f)$ amongst all maps in the same homotopy class. We now prove a new composition law:

Proposition 1.3.9

Let M,N be Riemann surfaces and let P be a smooth Riemannian manifold of any dimension. Let $f:M\to N$ be holomorphic or antiholomorphic and let $k:N\to P$ be a smooth map. Then: (1) if k is harmonic then $k\circ f$ is harmonic, (2) Conversely, if f is surjective, then if $k\circ f$ is harmonic, k is harmonic.

Note

By (1.3.7) it is not necessary to specify metrics on M or N.

Proof

Choose any hermitian metrics g,h for M,N. Assume f is holomorphic or antiholomorphic. By the composition law (1.2B), at a point $p \in M$:

(1.3F) $\quad \tau(k \circ f)(p) = dk \circ \tau(f)(p) + \mathrm{Tr}\,\nabla dk(df,df)(p)$.

Now, since f is holomorphic or antiholomorphic and therefore harmonic,

(1.3G) $\quad \tau(f)(p) = 0 \;\forall p \in M$;

also by (1.2D),

(1.3H) $\quad \mathrm{Tr}\,\nabla dk(df,df)(p) = \sum_{i=1}^{2} \nabla dk(df(e_i),df(e_i))$

where $\{e_i\}$ is an orthonormal basis for TM_p . We now claim:

there exists an orthonormal basis (e'_1,e'_2) for $TN_{f(p)}$ such that:

(1.3I) $\quad df(e_i) = \lambda(p)e'_i \quad (i=1,2)$ where $\lambda(p) \geq 0$.

For since f is holomorphic or antiholomorphic, either $df(p) = 0$ in which case any orthonormal basis will do, or $df(p) \neq 0$ in which case f is conformal at p; therefore since e_1,e_2 are orthogonal and of equal norm, so are $df(e_1),df(e_2)$, thus $df(e_1)$, $df(e_2)$ can be written in the claimed form (1.3I) with

$$\lambda(p) = \|df(e_1)\|_N = \|df(e_2)\|_N$$

(Here $\| \; \|_N$ as usual denotes the norm on $TN_{f(p)}$ determined by the metric h.)

Now

$$\lambda(p)^2 = \|df(e_1)\|_N^2 = \|df(e_2)\|_N^2 = \tfrac{1}{2}\left\{\|df(e_1)\|_N^2 + \|df(e_2)\|_N^2\right\}$$

Thus

$$\lambda(p)^2 = \tfrac{1}{2}\|df\|^2(p)$$

using (1.1K), where $\| \; \|$ as usual denotes the norm on $L(TM_p,TN_{f(p)})$. Thus from (1.3F) using (1.3G):

$$\begin{aligned}
\tau(k \circ f)(p) &= \mathrm{Tr}\,\nabla dk(df,df)(p) \\
&= \sum_{i=1}^{2} \nabla dk(df(e_i),df(e_i)) && \text{by (1.3H)} \\
&= \sum \nabla dk(\lambda(p)e'_i,\lambda(p)e'_i) && \text{by (1.3I)} \\
&= \lambda(p)^2 \sum \nabla dk(e'_i,e'_i) && \text{by bilinearity}
\end{aligned}$$

(1.3K) $\quad \tau(k \circ f)(p) = \tfrac{1}{2}\|df\|^2(p).\tau(k)(f(p)) \quad \forall p \in M \qquad$ by (1.3J).

We now prove part 1 of the proposition:

If k is harmonic, $\tau(k) \equiv 0$ and thus from (1.3J), $\tau(k \circ f) \equiv 0$, therefore $k \circ f$ is harmonic.

Proof of part 2 of the proposition:

If $k \circ f$ is harmonic, we have from (1.3J):

$$\|df\|^2(p) . \tau(k)(f(p)) = 0 \quad \forall p \in M. \text{ Therefore,}$$

(1.3L) $$\text{for each } p \in M: \begin{cases} \text{either} & \|df\|^2(p) = 0, \text{ i.e. } df(p) = 0 \\ \text{or} & \tau(k)(f(p)) = 0. \end{cases}$$

Now the set of regular values of $f: M \to N$ is the set of points $b \in N$ such that $\forall p \in f^{-1}(b)$, $df(p) \neq 0$. If f is not surjective, $f^{-1}(b)$ may be empty. However, if f is surjective, given a regular value b we can choose at least one $p \in M$ such that $b = f(p)$ and $df(p) \neq 0$. By (1.3L) it follows that $\tau(k)(b) = 0$. Therefore $\tau(k)$ is zero on the set of regular values of f. But by the theorem of Sard & Brown, the set of regular values is dense in N, therefore $\tau(k)$ is zero on a dense subset of N, and thus by continuity is zero on the whole of N, i.e. k is harmonic.∎

Remarks

(1) The previously known composition law required f to be a conformal diffeomorphism [E-S p.126].

(2) The result does not appear to generalise to compositions with holomorphic maps between Kähler manifolds despite such maps being harmonic [E-S p.118]. Such maps do not have the required property of conformality used in our proof.

D Harmonic maps and a quadratic differential

Let M,N be Riemann surfaces equipped with hermitian metrics g,h and local complex coordinates $(x,y),(u,v)$. We shall use complex notation $(z,\bar{z})$, $(w,\bar{w})$, where $z=x+iy$, $w=u+iv$. The metrics on M,N can be written:

$$g = \rho^2(x,y)\ (dx^2+dy^2) = \rho^2(z)\ dzd\bar{z}$$
$$h = \sigma^2(u,v)\ (du^2+dv^2) = \sigma^2(w)\ dwd\bar{w}$$

where $dz=dx+idy$, $d\bar{z}=dx-idy$ etc. Let $f:M\to N$ be a smooth map. A staitforward calculation (c.f. [E-E p.138]) yields the following expression for $\tau(f)$ in the local complex coordinates z,w:

(1.3M) $$\tau(f) = \frac{4}{\rho^2}\left\{ w_{z\bar{z}} + 2\frac{\sigma_w}{\sigma} w_z w_{\bar{z}} \right\}$$

Here subscripts denote partial derivatives, e.g. $w_{\bar{z}} = \frac{\partial w}{\partial \bar{z}}$ (see also §1.3A)

Remark

$$\frac{4}{\rho^2} w_{z\bar{z}} = \frac{1}{\rho^2}\left\{\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right\}$$ is the Laplacian of w. If f is holomorphic (resp. antiholomorphic), $w_{\bar{z}} = 0$ (resp. $w_z = 0$) hence $w_{z\bar{z}} = 0$ hence $\tau(f) = 0$, thus we get the Eells-Sampson result that a holomorphic or antiholomorphic map is harmonic.∎

We shall now use (1.3M) to link up our concept of harmonic with a concept of harmonic expounded by Gerstenhaber and Rauch [G-R] and Shibita [Sh]. We write several expressions for the pull-back of the metric on N:

$$\begin{aligned} f^*h &= \sigma^2(w) dw d\bar{w} \\ &= \sigma^2(w(z))\left\{\frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial \bar{z}}d\bar{z}\right\}\left\{\frac{\partial \bar{w}}{\partial \bar{z}}dz + \frac{\partial \bar{w}}{\partial z}d\bar{z}\right\}\end{aligned}$$

Multiplying out and using $\frac{\partial \bar{w}}{\partial z} = \overline{\frac{\partial w}{\partial \bar{z}}}$ etc.:

$$f^*h = 2\,\mathrm{Re}\,(a(z)dz^2) + b(z)dzd\bar{z}$$

where

(1.3N) $$a(z) = \sigma^2(w(z))\ \frac{\partial w}{\partial z}\overline{\frac{\partial w}{\partial \bar{z}}}$$

$$b(z) = \sigma^2(w(z)\left\{\left|\frac{\partial w}{\partial z}\right|^2 + \left|\frac{\partial w}{\partial \bar{z}}\right|^2\right\}$$

If we use different local complex coordinates (x',y') so that:

$$f^*h = 2\mathrm{Re}(a'(z')dz'^2) + b'(z')dz'd\bar{z}'$$

then it is easy to show:

$$(1.30) \qquad a'(z') = a(z)\left(\frac{dz}{dz'}\right)^2$$

thus the expression $a(z)dz^2$ is invariant under change of local complex coordinates; it is called a quadratic differential.

Lemma 1.3.10

If $df(p) = 0$, then $a(z)dz^2 = 0$ at p.

Proof

If $df(p) = 0$, $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial \bar{z}} = 0$ at p, therefore from (1.3N),

$a(z)dz^2 = 0$ at p.∎

Proposition 1.3.11

Let $f:M \to N$ be a smooth mapping between connected Riemann surfaces such that the zero set Σ_2 of df does not disconnect M i.e. $M \setminus \Sigma_2$ is connected. Then the quadratic differential $a(z)dz^2 \equiv 0$ if and only if f is holomorphic or antiholomorphic.

Proof

'if' If f is holomorphic (resp. antiholomorphic) $\frac{\partial w}{\partial \bar{z}} \equiv 0$ (resp. $\frac{\partial w}{\partial z} \equiv 0$), then from (1.3N), $a(z)dz^2 \equiv 0$.

'only if' Suppose $a(z)dz^2 \equiv 0$, then from (1.3N), since $\sigma^2 \neq 0$,

$\frac{\partial w}{\partial z}(p) \cdot \frac{\partial w}{\partial \bar{z}}(p) = 0$ for all $p \in M$. We must show that this implies:

$\frac{\partial w}{\partial z} \equiv 0$ or $\frac{\partial w}{\partial \bar{z}} \equiv 0$.

Let the set of points $p \in M$ such that $\frac{\partial w}{\partial \bar{z}}(p) = 0$, $\frac{\partial w}{\partial z}(p) = 0$ be Z_1, Z_2 respectively. Then:

(1) Z_1, Z_2 are closed sets and $M = Z_1 \cup Z_2$,

(2) M is the disjoint union: $(M \setminus Z_1) \cup (M \setminus Z_2) \cup (Z_1 \cap Z_2)$,

(3) $Z_1 \cap Z_2 = \Sigma_0$ by (1.3.10), since $p \in Z_1 \cap Z_2 \Leftrightarrow \frac{\partial w}{\partial z}(p) = 0$ and $\frac{\partial w}{\partial \bar{z}}(p) = 0$

$\Leftrightarrow df(p) = 0$,

(4) $M\setminus(Z_1\cup Z_2)$ is the disjoint union: $(M\setminus Z_1)\cup(M\setminus Z_2)$ of open sets.

Now by hypothesis, $M\setminus\Sigma_0$ is connected, therefore by (3) above $M\setminus(Z_1\cap Z_2)$ is connected, therefore from (4) above, one of the sets $M\setminus Z_2$, $M\setminus Z_1$ must be empty. Therefore by (1) above, $Z_1=M$ or $Z_2=M$, i.e. $\frac{\partial w}{\partial \bar{z}} \equiv 0$ or $\frac{\partial w}{\partial z} \equiv 0$, i.e. f is holomorphic or antiholomorphic.∎

For harmonic maps we can improve this result:

Corollary 1.3.12

Let $f:M\to N$ be a mapping between Riemann surfaces which is harmonic with respect to some choice of hermitian metric on N. Then the Quadratic differential $a(z)dz^2$ is identically zero on M if and only if f is holomorphic or antiholomorphic.

Proof

We must appeal to a result to be proved independently in §2.2 that the zeros of df are isolated. Then clearly the zero set Σ_0 of df does not disconnect M. We may thus apply (1.3.10).

We now ask what properties $a(z)dz^2$ has for a harmonic mapping. We call the quadratic differential $a(z)dz^2$ holomorphic iff $a(z)$ is holomorphic function of the local complex coordinate z. By (1.30) this definition is independent of the local complex coordinates chosen. We now determine when $a(z)dz^2$ is holomorphic by computing $\frac{\partial}{\partial \bar{z}}a(z)$; differentiating (1.3N) with respect to $\bar{z}$ and using $\overline{\frac{\partial w}{\partial \bar{z}}} = \frac{\partial \bar{w}}{\partial z}$ etc.:

$$\frac{\partial}{\partial \bar{z}}a(z) = 2\sigma(\sigma_w w_{\bar{z}} + \sigma_{\bar{w}}\bar{w}_{\bar{z}})w_z\bar{w}_{\bar{z}} + \sigma^2(w_{z\bar{z}}\bar{w}_z + w_z\bar{w}_{z\bar{z}}) \cdot$$

Using (1.3M) this can be written:

$$(1.3P) \qquad \frac{\partial}{\partial \bar{z}}a(z) = \frac{\sigma^2\rho^2}{4}\left\{\tau(f)\bar{w}_{\bar{z}} + \overline{\tau(f)}w_z\right\}.$$

We say $f:M\to N$ is non-singular at $p\in M$ if its Jacobian $J(p)$ is non-zero. (Jacobian = Jacobian determinant - see list of symbols.)

Theorem 1.3.13

Let $f:M\to N$ be a smooth mapping between Riemann surfaces. Let N have a chosen hermitian metric h. Then:

(1) if f is harmonic with respect to h, $a(z)dz^2$ is holomorphic;

(2) conversely, if $a(z)dz^2$ is holomorphic and f is non-singular on a dense subset of M, f is harmonic.

Proof

(1) If f is harmonic, then $\tau(f)\equiv 0$, therefore from (1.3P), $\frac{\partial}{\partial\bar z}a(z)\equiv 0$, and thus $a(z)dz^2$ is holomorphic.

(2) We solve (1.3P) for $\tau(f)$ by computing:

$$w_{\bar z}\frac{\partial}{\partial\bar z}a(z) - w_z\overline{\frac{\partial}{\partial\bar z}a(z)} = -\frac{\sigma^2\rho^2}{4}(w_z\bar w_{\bar z} - w_{\bar z}\bar w_z).\ \tau(f)$$

Now a simple computation shows $w_z\bar w_{\bar z} - w_{\bar z}\bar w_z$ is the expression for the Jacobian J of f in complex notation, thus:

$$(1.3Q)\qquad -\frac{\sigma^2\rho^2}{4}.J.\ \tau(f) = w_{\bar z}\frac{\partial}{\partial\bar z}a(z) - w_z\overline{\frac{\partial}{\partial\bar z}a(z)}$$

Thus if $a(z)dz^2$ is holomorphic, the right-hand side of (1.3Q) is zero, therefore $J(p).\ \tau(f)(p) = 0$ for all $p\in M$. Thus if $J(p)$ is non-zero on a dense subset of M, we must have $\tau(f)(p) = 0$ for all p on the same dense subset, and thus,by continuity, $\tau(f)(p) = 0$ for all $p\in M$, i.e. f is harmonic.∎

Corollary 1.3.14

Let M,N be Riemann surfaces with M compact and of Euler characteristic $\chi(M)$, and let $f:M\to N$ be harmonic with respect to some hermitian metric on N but not holomorphic or antiholomorphic. Then $df(p) = 0$ at at most $-\chi(M)$ points $p\in M$. (*)

Proof

By (1.3.12) $a(z)dz^2$ is a holomorphic quadratic differential on M. Now it is well-known that a holomorphic quadratic differential is either identically zero or has at most $-\chi(M)$ zeros. In the former case, by

(*) Recall that $-(M) = 2g-2$ where $g = \text{genus}(M)$.

(1.3.11) f is holomorphic or antiholomorphic, in the latter case, by (1.3.10), every zero of df is a zero of $a(z)dz^2$, therefore $df(p) = 0$ at at most $-\chi(M)$ points $p \in M$. ∎

Corollary 1.3.15

Let $f: S^2 \to N$ be a harmonic map from the Riemann sphere into any Riemann surface equipped with a hermitian metric. Then f must be holomorphic or antiholomorphic.

Proof

Since any holomorphic quadratic differential on the Riemann sphere is identically zero, [G-Rp808], this follows from (1.3.10). ∎

Remark

It can further be shown that unless f is constant, it must be surjective, and N must be (conformally equivalent to) the Riemann sphere. ∎

Corollary 1.3.16

Let $f: T \to N$ be a harmonic map from any torus (i.e. compact Riemann surface of genus 1) into a Riemann surface N equipped with a hermitian metric. Then df can have no zeros unless f is holomorphic or antiholomorphic.

Proof

By (1.3.13) $df(p) = 0$ at at most 0 points unless f is holomorphic or antiholomorphic. ∎

Remark 1.3.17

(1.3.13) also holds in the non-orientable case: i.e. let M, N be smooth connected (not necessarily orientable) Riemannian surfaces, M compact; let $f: M \to N$ be harmonic. Then we can lift f to a mapping between orientable double covers: $\tilde{f}: \tilde{M} \to \tilde{N}$. Then if f is not holomorphic or antiholomorphic, df has at most $-\chi(M)$ zeros. Proof: by (1.3.13), $d\tilde{f}$ has at most $-\chi(\tilde{M})$ zeros. Now each zero of df is covered by precisely two zeros of $d\tilde{f}$ also $\chi(\tilde{M}) = 2\chi(M)$, therefore df has at most $\frac{1}{2}\chi(\tilde{M}) = \chi(M)$ zeros.

1.4 Quasi-analyticity of a smooth harmonic map.

A Introduction and preliminaries

Let M,N be connected smooth Riemannian manifolds and let $f:M\to N$ be a harmonic map. The harmonic equations (1.1F) are a quasi-linear elliptic system of a type studied by many authors, the basic theme being that if the various quantities g^{ij}, Γ_{ij}^{k}, $L_{\alpha\beta}^{\gamma}$ are C^{∞} functions of their variables, then although we know (c.f. (1.1.4)) that any C^2 solution f of $\tau(f) = 0$ is C^{∞}, such a solution behaves in some ways as if it were C^{ω}. A large number of papers have arisen from the basic ideas of Carlemann [Ca], mainly in the direction of generalising the classical Cauchy-Kowalewsky theorem to the C^{∞} case; for our purposes we shall give two results: a unique continuation property for harmonic maps derived from results of Aronszayn [Ar], and a local expansion of a harmonic map as a harmonic polynomial + higher order terms,derived from results of Hartman & Wintner [H-W1].

o notation

Let $u(q) = u(x^1,\dots,x^m)$ be a real- or complex-valued function of points $q = (x^1,\dots,x^m)$ on a domain $D\subset \underline{R}^m$, andlet $p=(x_0^{\,1},\dots,x_0^{\,m})$ be a fixed point of D. We write $|q-p|$ for the Euclidean distance between p and q, i.e:

$$|q-p| = \sqrt{\left\{\sum_{i=1}^{m}(x^i-x_0^{\,i})^2\right\}}$$

and we write

$$\rho = |q| = \sqrt{\sum_{i=1}^{m}(x^i)^2}$$

We say $u(q) = o(q-p)^k$ iff $\dfrac{u(q)-u(p)}{|q-p|^k}\to 0$ as $q\to p$. If $p=(0,\dots,0)$, we write $u(q) = o(q)^k$ or simply:

$$u = o(\rho)^k .$$

O notation

We write $u(q) = O(q-p)^l$ iff $\dfrac{|u(q)-u(p)|}{|q-p|^l}$ is bounded as $q\to p$.

Again if $p=(0,\dots,0)$, we write $u(q)=O(q)^l$ or simply $u = O(\rho)^l$.

We shall use the following properties of o,O:

Proposition 1.4.1

If $u = O(\rho)^l$ and $v = o(\rho)^k$ then $u.v = o(\rho)^{l+k}$

Proof

Trivial.∎

Now let $u(x^1,\dots,x^m)$ be C^∞. Then [W p.393] u has a Taylor expansion with remainder about $(0,\dots,0)$:

$$(1.4A)\qquad u(x^1,\dots,x^m) = \sum_{\sigma_\lambda < s} a_{\lambda_1\dots\lambda_m} \frac{(x^1)^{\lambda_1}\dots(x^m)^{\lambda_m}}{\lambda_1!\dots\lambda_m!} + \sum_{\sigma_\lambda = s} \frac{(x^1)^{\lambda_1}\dots(x^m)^{\lambda_m}}{\lambda_1!\dots\lambda_m!} F_{\lambda_1\dots\lambda_m}(x^1,\dots,x^m)$$

for any $s=0,1,\dots,$ where:

$$(1.4B)\qquad a_{\lambda_1\dots\lambda_m} = \frac{\partial^{\sigma_\lambda}}{\partial(x^1)^{\lambda_1}\dots\partial(x^m)^{\lambda_m}} u(0,\dots,0)\,, \qquad \sigma_\lambda = \sum_{i=1}^{m} \lambda_i$$

and the $F_{\lambda_1\dots\lambda_m}$ are C^∞ functions of $(x^1,\dots,x^m)$ with

$$(1.4C)\qquad F_{\lambda_1\dots\lambda_m}(0,\dots,0) = \frac{\partial^{\sigma_\lambda}}{\partial(x^1)^{\lambda_1}\dots\partial(x^m)^{\lambda_m}} u(0,\dots,0)\,.$$

Proposition 1.4.2

(1) If $u(x^1,\dots,x^m)$ is C^∞, then the following are equivalent:

(a) $u = o(\rho)^{l-1}$,

(b) $u = O(\rho)^l$,

(c) the partial derivatives of u of all orders $\leqslant l-1$ vanish at $(0,\dots,0)$,

(d) the Taylor series of u starts with terms in $(x^1)^{\lambda_1}\dots(x^m)^{\lambda_m}$ where $\lambda_1+\dots+\lambda_m \geqslant l$.

(2) If u is C^∞ and $u=o(\rho)^{l-1}$, then $\frac{\partial u}{\partial x^i}$ is C^∞ and $\frac{\partial u}{\partial x^i} = o(\rho)^{l-2}$.

Proof of part (1)

(c)⇔(d) Clear from (1.4A) and (1.4B).

(c)$\Rightarrow$(b): For by (1.4A) with s=l,

$$u(x^1,\dots,x^m) = \sum_{\sigma_\lambda = \ell} \frac{(x^1)^{\lambda_1}\dots(x^m)^{\lambda_m}}{\lambda_1!\dots\lambda_m!} F_{\lambda_1\dots\lambda_m}(x^1,\dots,x^m) .$$

From this it is easily seen that $u=O(\rho)^l$.

(b)$\Rightarrow$(a): Immediate from definitions.

(a)$\Rightarrow$(d): For if Taylor series starts with terms in $(x^1)^{\lambda_1}\dots(x^m)^{\lambda_m}$ where $\lambda_1+\dots+\lambda_m = s < l$, from (1.4A) we can write:

$$u(x^1,\dots,x^m) = \sum_{\sigma_\lambda = s} \frac{(x^1)^{\lambda_1}\dots(x^m)^{\lambda_m}}{\lambda_1!\dots\lambda_m!} F_{\lambda_1\dots\lambda_m}(x^1,\dots,x^m).$$

where $F_{\lambda_1\dots\lambda_m}(0,\dots,0) \neq 0$. From this it is seen that u is not $=o(\rho)^{l-1}$

Putting the above implications together, we have (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d).

Proof of part (2)

If u is C^∞ and $=o(\rho)^{l-1}$, then by part 1, the partial derivatives of u of all orders $\leq$ l-1 vanish at $(0,\dots,0)$, therefore the partial derivatives of $\frac{\partial u}{\partial x^i}$ of all orders $\leq$ l-2 vanish at $(0,\dots,0)$. Hence by part (1) above, $\frac{\partial u}{\partial x^i} = o(\rho)^{l-1}$ $(i=1,\dots,m)$ ∎

B The Unique Continuation Theorem

Theorem 1.4.3 (Aronszayn) [Ar p.248]

Let $u^1,\dots,u^n$ be smooth functions of points $q = (x^1,\dots,x^m)$ on a domain $D\subset \underline{R}^m$ which satisfy the differential inequality:

$$(1.4D) \qquad |Lu^k(q)|^2 \leq \text{const.} \sum_{\alpha=1}^{n}\left\{\sum_{i=1}^{m}\left|\frac{\partial u^\alpha}{\partial x^i}\right|^2 + |u^\alpha|^2\right\}$$

where L is a smooth linear elliptic differential operator of second order. Suppose at some point $p\in D$, the n-tuple $(u^1,\dots,u^n)$ has a zero of infinite order, i.e: $u^\alpha(q) = o(q-p)^k$ $\forall k=1,2,\dots, \forall\alpha=1,\dots,n$.

Then u is identically zero on its domain. ∎

Remarks

Aronszayn requires less stringent conditions: (1) on the smoothness of the u^α and L, (2) the n-tuple $(u^1,\ldots,u^n)$ is required only to have a "zero of infinite order in the 1-mean" (see [Ar]). Even better results are contained in a later paper [A-K-S]. ∎

We now use Aronszayn's result to prove:

Theorem 1.4.4 (Sampson) Unique Continuation Theorem [Sa]

Let M,N be connected smooth Riemannian manifolds, and let $f,f':D\to N$ be two harmonic maps defined on a domain $D\subset M$. Then if $f,f':D\to N$ "agree to infinitely high order" at some point $p\in D$, i.e:

$$f^\alpha(q) - f'^\alpha(q) = o(q-p)^k \quad \forall k=1,2,\ldots, \forall\alpha=1,\ldots,n,$$

then they agree on their whole domain, i.e. $f(q)=f'(q)$ for all $q\in D$.

Corollary 1.4.5

If f,f' agree on an open set, they agree on their whole domain.

Proof of Theorem 1.4.4 [Sa]

Take local coordinates $(x^1,\ldots,x^n)$ near p, $(u^1,\ldots,u^n)$ near f(p), let the point with coordinates $(x^1,\ldots,x^m)$ be denoted by q. Consider the difference $w^\gamma = f^\gamma - f'^\gamma$. Subtracting the harmonic equation (1.1F) for f' from that for f we get:

$$\Delta w^\gamma(q) = g^{ij}\left\{ L_{\alpha\beta}^{\gamma}(f(q))f_i^\alpha f_j^\beta - L_{\alpha\beta}^{\gamma}(f'(q))f'^\alpha_i f'^\beta_j \right\}$$

for all q in some neighbourhood of p. This can be written

$$(1.4E)\qquad \Delta w^\gamma(q) = L_{\alpha\beta}^{\gamma}(f(q))g^{ij}\quad (w_i^\alpha f_j^\beta - f'^\alpha_i w_j^\beta) \\ + (L_{\alpha\beta}^{\gamma}(f(q)) - L_{\alpha\beta}^{\gamma}(f'(q)))f'^\alpha_i f'^\beta_j g^{ij}$$

Consider the second term of the right-hand side of (1.4E). Since $L_{\alpha\beta}^{\gamma}$ is differentiable,

$$\left| L_{\alpha\beta}^{\gamma}(f(q)) - L_{\alpha\beta}^{\gamma}(f'(q))\right| \leq \text{const.}\sum_{a=1}^{n}\left|f^a(q)-f'^a(q)\right| = \text{const.}\sum_{a=1}^{n}\left|w^a\right|$$

It is now clear from (1.4E) that

$$|\Delta w^{\gamma}|^2 \leq \text{const}\sum_{\alpha=1}^{n}\left\{\sum_{i=1}^{m}\left|\frac{\partial w^{\alpha}}{\partial x^{i}}\right|^2 + |w^{\alpha}|^2\right\} \text{ in a neighbourhood of p.}$$

Further, by hypothesis, $w^{\gamma}(q) = f^{\gamma}(q)-f'^{\gamma}(q)$ has a zero of infinite order at p, therefore by(1.4.3), f,f' agree on a coordinate neighbourhood of p, therefore,by a simple argument, f,f' agree on their whole domain D. ∎

Remark

Thus a harmonic map is determined by its values on an arbitrary open set.

C Expansion of a harmonic map between surfaces

We first give the result which we shall use to expand a harmonic map as harmonic polynomial + higher order terms. By an expression such as $u = u_0 + o(\rho)^{l-1}$ we shall mean $u-u_0 = o(\rho)^{l-1}$.

Theorem 1.4.6 (Hartman & Wintner HW theorem 1* p.461 and remarks p.450)

Let g^{11}, g^{12}, g^{22} be C^1 functions of (x,y) defined on an open disk $D\subset\underline{\mathbb{R}}^2$,centre $(0,0)$, such that $g^{11}(0,0) = g^{22}(0,0) = 1$, $g^{12}(0,0) = 0$, and let $H(x,y,u,p,q)$ be a continuous function of its five real variables such that:

(1.4F) for every $\varepsilon>0$, $M>0$, there exists a constant $K=K(\varepsilon,M)$ such that:

$$|H(x,y,u,p,q)| \leq K(|u|+|p|+|q|) \text{ whenever } x^2+y^2<\varepsilon^2,\ |u|\leq M,\ |p|\leq M,\ |q|\leq M.$$

Let $u(x,y)$ be a C^2 solution of:

$$g^{11}\frac{\partial^2 u}{\partial x^2} + 2g^{12}\frac{\partial^2 u}{\partial x\partial y} + g^{22}\frac{\partial^2 u}{\partial y^2} + H(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = 0$$

with $u(0,0) = 0$.

Then:

(1) $u=u(x,y)$ is not $=o(\rho)^n\ \forall n$ (where $\rho = \sqrt{x^2+y^2}$) unless u is identically 0,

(2) either $u\equiv 0$ or, writing $z=x+iy$, u admits an expansion:

(1.4G) $u = re(a_l z^l) + o(\rho)^l$ where $l\in\{1,2,3,\ldots\}$, $a_l\in\underline{C}$, $a_l\neq 0$,

(Here re,im denote real,imaginary part of a complex number respectively) further, the partial derivatives $u_x = \frac{\partial u}{\partial x}$, $u_y = -\frac{\partial u}{\partial y}$ are given by:

$$(1.4H) \qquad \begin{aligned} u_x &= re(la_l z^{l-1}) + o(\rho)^{l-1} \\ u_y &= -im(la_l z^{l-1}) + o(\rho)^{l-1} \end{aligned} \qquad \blacksquare$$

Remarks

(1) If u is smooth (1.4G) exhibits a Taylor expansion with first term $re(a_l z^l)$ and remainder term $o(\rho)^l$. By (1.4.2) this term is equally well written $O(\rho)^{l+1}$. Further, if u is smooth, (1.4H) follows directly from (1.4G) by proposition (1.4.2) part 2.

(2) The term $re(a_l z^l)$ is a homogeneous harmonic polynomial in (x,y). $\blacksquare$

Now let M,N be Riemann surfaces with chosen hermitian metrics g,h, and let $f:M \to N$ be harmonic (with respect to h). Let $p \in M$ and take local coordinates (x,y), (u,v) centred on p, f(p) respectively. Then in these coordinates f is given by a pair of functions: $u=u(x,y)$, $v=v(x,y)$. We shall now see that if we chose the coordinates in a special way we may apply the above theorem to get expansions of the type (1.4G) for u and v. In fact, let (x,y) be isothermal at p , then by (1.3.3), $g^{11}(0,0) = g^{22}(0,0)$ and $g^{12}(0,0) = 0$. By multiplication of the hermitian metric g by a suitable constant we can assume:

$$(1.4I) \qquad g^{11}(0,0) = g^{22}(0,0) = 1 \ , \quad g^{12}(0,0) = 0 \ .$$

Let (u,v) be normal coordinates centred on f(p) (see after (1.1.7)); recall q ($\in$ neighbourhood of p) has normal coordinates (u,v) iff $\exp_{f(p)}^{-1}(q) \in TM_{f(p)}$ has components (u,v) with respect to some basis for $TM_{f(p)}$ (which we are not requiring to be an orthonormal basis).

Lemma 1.4.7

In normal coordinates (u,v) centred on $b \in N$:

(a) the equations of all geodesics through b are linear; conversely any linear equation $Au+Bv=0$ (A,B constants not both zero) is the equation of a geodesic through b; in particular the axes $v=0$, $u=0$ are geodesics;

(b) the Christoffel symbols for N satisfy:

$$L_{11}^{\ 1}(u,0) = L_{11}^{\ 2}(u,0) = 0 \quad \forall u \ , \ L_{22}^{\ 1}(0,v) = L_{22}^{\ 2}(0,v) = 0 \quad \forall v \ ;$$

(c) if (u',v') are other local coordinates centred on b, (u',v') are normal if and only if the transformation of coordinates $(u,v) \mapsto (u',v')$ is linear.

Proof

(a) If $t \in TN_b$, then for all $s \in \mathbb{R}$, $\exp_b(st)$ lies on the geodesic through b tangent to t. Now if (u,v) satisfy the equation

(1.4J) $\quad Au+Bv=0$

then writing this equation in parametric form as:

$$(u,v) = s(-B,A) \qquad (\text{parameter } s)$$

shows that (u,v) lies on the geodesic through b tangent to $-B\frac{\partial}{\partial u} + A\frac{\partial}{\partial v}$ where $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}$ are the basis for the tangent space TN_b determined by the coordinates (u,v). Conversely, varying B and A gives all tangent directions, and thus all geodesics through b have an equation of the form (1.4J). Note that the axes $u=0, v=0$ are thus geodesics.

(b) The axis $v=0$ has parametric equations:

$$u=u, \ v=0 \qquad (\text{parameter } u)$$

Applying the differential equation for a geodesic [Hi p.58] to the geodesic $v=0$ parametrised as above shows $L_{11}^{\ 1}(u,0) = 0$, $L_{11}^{\ 2}(u,0) = 0$ $\forall u$. The other identities follow similarly.by considering the other axis.

(c) The coordinates (u,v) of a point q are the components of $(\exp_b)^{-1}(q)$ with repect to some basis for TN_b . Hence the change of coordinates: $(u,v) \mapsto (u',v')$ is linear if and only if $(u'v')$ are the coordinates of q with respect to some new basis for TN_b , i.e. if and only if (u',v') are normal coordinates of q with respect to some basis for TN_b . ∎

Theorem 1.4.8

Let M,N be Riemann surfaces; let N have a chosen hermitian metric.

Let $f:M\to N$ be harmonic with respect to this metric. Let $p\in M$ and let (x,y), (u,v) be local coordinates centred on p, $f(p)$ respectively, write $z=x+iy$, $\rho=|z|$. Then:

(A) if

$$(1.4K)\quad \begin{cases} (1) & (x,y) \text{ are isothermal at } p \quad (\text{see } (1.3.3)) \\ (2) & (u,v) \text{ are normal coordinates centred on } f(p) \end{cases}$$

then in some neighbourhood U of p, f has one of the forms:

$$(1.4L)\quad \begin{cases} (a) & u\equiv 0 \ , \ v\equiv 0 \\ (b) & u\equiv 0 \ , \ v = \mathrm{re}(b_k z^k) + o(\rho)^k \\ (b') & u = \mathrm{re}(a_l z^l) + o(\rho)^l \ , \ v\equiv 0 \\ (c) & u = \mathrm{re}(a_l z^l) + o(\rho)^l \ , \ v = \mathrm{re}(b_k z^k) + o(\rho)^k \end{cases}$$

where l,k are positive integers and a_l, b_k are non-zero complex numbers ;

(B) call the coordinates (x,y) , (u,v) <u>admissible</u> if they are of the type (1.4K) and also f is of the form (1.4L)(a), (1.4L)(b) or (1.4L)(c) with $l>k$ or $l=k$ and $\mathrm{im}(a_l\overline{b_k})\neq 0$; then admissible coordinates can be obtained from arbitrary coordinates of the type (1.4K) by a suitable linear transformation of (u,v) coordinates;

(C) if (x,y) , (u,v) are admissible coordinates, the the matrix for df and J (the derivative and Jacobian determinant of f) are given in the neighbourhood U of p as follows:

$$(1.4M)\quad \begin{cases} \text{(in case (a) of (1.4L))} & df\equiv 0 \ , \ J\equiv 0 \\ \text{(in case (b) of (1.4L))} & df = \begin{bmatrix} 0 & 0 \\ \mathrm{re}(kb_k z^{k-1})+o(\rho)^{k-1} & -\mathrm{im}(kb_k z^{k-1})+o(\rho)^{k-1} \end{bmatrix} \\ & J\equiv 0 \\ \begin{array}{l}\text{(in case (c)} \\ \text{of (1.4L)} \\ \text{with } l>k \\ \text{or } l=k \text{ and} \\ \mathrm{im}(a_l\overline{b_k})\neq 0\end{array} & \begin{array}{l} df = \begin{bmatrix} \mathrm{re}(la_l z^{l-1})+o(\rho)^{l-1} & -\mathrm{im}(la_l z^{l-1})+o(\rho)^{l-1} \\ \mathrm{re}(kb_k z^{k-1})+o(\rho)^{k-1} & -\mathrm{im}(kb_k z^{k-1})+o(\rho)^{k-1} \end{bmatrix} \\ J = lk\,|z|^{2k-2}.\mathrm{im}(a_l\overline{b_k}z^{l-k}) + o(\rho)^{l+k-2} \end{array} \end{cases}$$

<u>Notes</u>

The condition "$l>k$ or $l=k$ and $\mathrm{im}(a_l\overline{b_k})\neq 0$" can be interpreted as requiring the lowest terms $re(a_1 z^1)$, $re(b_k z^k)$ of u,v to be linearly independent in the vector space over $\underline{R}$ of smooth functions of (x,y).

Proof of theorem part (A)

Choose a hermitian metric g for M. We wish to apply theorem (1.4.6) to expand u(x,y), v(x,y) in the form (1.4G). Now, if (x,y) are isothermal at p, by (1.3.3) $g^{11}(0,0) = g^{22}(0,0)$ and $g^{12}(0,0) = 0$.

Now, since f is harmonic, by the harmonic equations (1.1F), u(x,y) satisfies:

$$\tau(f)^1 \equiv g^{11}\frac{\partial^2 u}{\partial x^2} + 2g^{12}\frac{\partial^2 u}{\partial x \partial y} + g^{22}\frac{\partial^2 u}{\partial y^2} + H(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = 0$$

where

$$H(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = -f_k{}^1 g^{ij}\Gamma_{ij}^{\ k} + g^{ij} f_i^{\alpha} f_j^{\beta} L_{\alpha\beta}{}^1$$

or in full, since $f_1{}^1 = \frac{\partial u}{\partial x}$ etc:

$$\begin{aligned} H(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = & -\frac{\partial u}{\partial x} g^{ij}\Gamma_{ij}^{\ 1} - \frac{\partial u}{\partial y} g^{ij}\Gamma_{ij}^{\ 2} \\ & + \left\{ g^{11}\left(\frac{\partial u}{\partial x}\right)^2 + 2g^{12}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + g^{22}\left(\frac{\partial u}{\partial y}\right)^2 \right\} L_{11}{}^1 \\ & + 2\left\{ g^{11}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2g^{12}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + g^{22}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right\} L_{12}{}^1 \\ & + \left\{ g^{11}\left(\frac{\partial v}{\partial x}\right)^2 + 2g^{12}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + g^{22}\left(\frac{\partial v}{\partial y}\right)^2 \right\} L_{22}{}^1 \end{aligned}$$

Now for any chosen $\varepsilon > 0$, $M > 0$, it is clear from this expression that if $x^2+y^2 < \varepsilon^2$, $|u| \le M$, $\left|\frac{\partial u}{\partial x}\right| \le M$, $\left|\frac{\partial u}{\partial y}\right| \le M$, there exist constants such that:

(1.4N) $\quad |H| \le \text{const.}\left\{ \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| \right\} + \text{const.}\left| L_{22}{}^1(u,v) \right|$

But $L_{22}{}^1(u,v)$ is a smooth function, which by (1.4.7)(b) is zero when u=0, therefore [W§14(g)], $L_{22}{}^1(u,v) = u \times \{\text{smooth funtion of } (u,v)\}$, in particular, for any chosen $\eta > 0$, there exists a constant such that

$$\left| L_{22}{}^1(u,v) \right| \le \text{const.}\,|u| \quad \forall (u,v) \text{ s.t. } u^2+v^2 < \eta^2.$$

Since (u,v) is a continuous function of (x,y), we deduce that, given $\varepsilon > 0, M > 0$ there exists constant such that whenever $x^2+y^2 < \varepsilon^2$, $|u| \le M$, $\left|\frac{\partial u}{\partial x}\right| \le M$, $\left|\frac{\partial u}{\partial y}\right| \le M$:

$$|H| \le \text{const.}\left\{ |u| + \left|\frac{\partial u}{\partial x}\right| + \left|\frac{\partial u}{\partial y}\right| \right\}$$

We can now apply theorem (1.4.6) to show:

$$\text{either} \quad u \equiv 0 \quad \text{or} \quad u = re(a_1 z^1) + o(\rho)^1$$

where l is a positive integer, a_1 a non-zero complex number.

Similarly it can be shown:

$$\text{either} \quad v \equiv 0 \quad \text{or} \quad v = re(b_k z^k) + o(\rho)^k$$

where k is a positive integer, b_k a non-zero complex number.

Combining these results gives the four cases (a),(b),(b'),(c) of (1.4L).

Proof of part (B)

We have just shown that in any coordinate system of the type (1.4K) f has the form (a),(b),(b') or (c) of (1.4L). If f has the form (b'), performing the linear transformation of coordinates $(u',v') = (v,u)$ and dropping dashes converts f into the form (b). If f has the form (c) with $l<k$, the same transformation of coordinates converts f into the form (c) with $l>k$. If f has the form (c) with $l=k$ and $im(a_1\overline{b_k})=0$, perform the linear transformation of coordinates:

$$u' = u - b_k^{-1} a_1 v \ , \ v' = v \ .$$

(Note that $b_k^{-1}a_1$ is real by the equation $im(a_1\overline{b_k})=0$.) Dropping dashes we see that f is now of the form (b) or (c) with $l>k$.

Thus we have shown that by a linear change of (u,v) coordinates we can assume f is of the form (1.4L)(a), (1.4L)(b) or (1.4L)(c) with $l>k$ or $l=k$ and $im(a_1\overline{b_k})\neq 0$. By (1.4.7)(c) the new (u,v) coordinates are normal, by definition, they are admissible.

Proof of part (C)

Let (x,y) , (u,v) be admissible coordinates. Then f is of the form (1.4L)(a), (1.4L)(b) or (1.4L)(c) with $l>k$ or $l=k$ and $im(a_1\overline{b_k})\neq 0$. Straitforward calculations using (1.4.2)(2) and (1.4.1) yield the expressions (1.4M). Note that in (1.4M)(c), if $l=k$, the expression for J reduces to:

$$J = lk|z|^{2k-2}.im(a_1\overline{b_k}) + o(\rho)^{2k-2}$$

■

Remarks

(1). It can easily be shown that, with the conventions that $l=k=\infty$ in case (a) of (1.4L) and $l=\infty$ in case (b), the integers l and k are invariants whatever particular admissible coordinates are chosen.

(2) Compare the theorem with work of Heinz [He1] who gives the expansions (1.4L) for a solution $u(x,y)$, $v(x,y)$ of a system of quasi-linear elliptic partial differential equations of the same type as the harmonic equations (1.1F). However, it is not possible, in Heinz' case, to give a nice description of the coordinates required such as (1.4K). In the C^{ω} case Heinz' expansions are given by Lewy [Le1].

(3) The result is essentially two-dimensional; although the result (1.4.6) of Hartman & Wintner can be generalised to higher dimensions (see [H-W2]), to apply this result we must impose severe restrictions on M,N, (we can apply the result when M and N are flat).

2. SINGULARITIES OF HARMONIC MAPS BETWEEN SURFACES

In this chapter we classify the singularities that a harmonic map between surfaces can have into four basic types. In §2.1, we shall define and discuss the relevant types of singularities for a smooth mapping between surfaces; in §2.2, we classify the singularities for a harmonic map between surfaces; in §2.3, we compare with the singularities possible for an arbitrary smooth map; in §2.4, we compare with the harmonic maps in higher dimensions and show our results are essentially 2-dimensional.

2.1 Singularities of a smooth mapping

A Critical set and critical values

Let M,N be connected smooth manifolds of dimension m,n respectively, and let $f:M\to N$ be a smooth map. (Occasionally, we shall suppose that M,N,f are real-analytic.) The derivative of f is a smooth linear bundle map $df:TM\to TN$; by the *rank of f at $p\in M$* or the *rank of df at p* we mean the rank of the linear mapping $df(p)$: $TM_p\to TN_{f(p)}$. The set of *critical points* or *critical set* Σ of f is the set of points $p\in M$ such that $\text{rank}\, df(p)<\min(m,n)$. Points of M not in the critical set are called *ordinary points* . Points of $f(\Sigma)$ are called *critical values* and points of $N\setminus f(\Sigma)$ are called *regular values* . The Sard-Brown theorem tells us that the set of regular values is dense in N.

Suppose now that $\dim M = \dim N = n$. Critical points, critical values are now termed *singular points*, *singular values* respectively and Σ is termed the *singular set*. Take local coordinates $(x_1,\dots,x_n)$, $(u_1,\dots,u_n)$ for M,N, then at each point $p\in M$, we may calculate the Jacobian determinant $J(p)=\dfrac{\partial(u_1,\dots,u_n)}{\partial(x_1,\dots,x_n)}(p)$ of f; the singular set Σ is the set of points $p\in M$ at which the Jacobian determinant vanishes. By the inverse function theorem, Σ is the set on which f fails to be a local diffeomorphism.

We may also define the branch set B of $f:M\to N$, this is the set on which f is fails to be a local homeomorphism , $f(B)$ is called the set of branch values of f , note $B\subset\Sigma$. Now define:

$$\Sigma_i = \{ p\in M: \text{rank } df(p) = n-i \}$$

We see that M is the disjoint union: $\Sigma_0 \cup \dots \cup \Sigma_n$. Further, Σ_0 = set of ordinary points of M, and $\Sigma_1 \cup \dots \cup \Sigma_n = \Sigma$.

Now suppose $\dim M = \dim N = 2$. Then M is the disjoint union: $\Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ and further:

$\Sigma_0 = \{ p\in M: J(p)\neq 0 \}$ = set of ordinary points = set of points where f is a local diffeomorphism ;

$\Sigma_1 = \{ p\in M: J(p)=0 \text{ but } df(p)\neq 0 \}$;

$\Sigma_2 = \{ p\in M: df(p)=0 \}$;

Singular set $\Sigma = \Sigma_1 \cup \Sigma_2$.

B 1-submanifolds, endpoints and meeting points

Let M be a connected smooth surface. A C^r 1-submanifold γ of M $(r=0,1,\ldots,\infty)$ is a subset $\gamma\subset M$ such that for each $p\in\gamma$, there exists a neighbourhood U of p in M, neighbourhoods V,V' of 0 in $\underline{\mathbb{R}}$ and C^r coordinates (t,t') on $U\subset M$, centred on p, such that:

$$U=\{(t,t')\in V\times V'\},\quad \gamma\cap U=\{(t,t')\in V\times\{0\}\}\qquad (\dagger)$$

For any such C^r coordinates (t,t'), we shall call t a C^r (local) parameter for γ (centred on p), note that the C^r mapping $\tilde{\gamma}:V\to M$, $t\mapsto(t,0)$ parametrises $\gamma\cap U$ as a C^r arc of M (c.f. §1.1C). For $r>0$, the tangent to γ at p can be found as the vector $\frac{d\tilde{\gamma}}{dt}(0)\in TM_p$. This is well-defined up to scalar multiples under change of local C^r parameter t $(r>0)$.

Endpoints:

We shall say p is an endpoint of class C^s or C^s endpoint $(s=0,1,\ldots,r)$ of the C^r 1-submanifold iff there exists a neighbourhood U of p in M, neighbourhoods V,V' of 0 in $\underline{\mathbb{R}}$ and C^r coordinates for U centred on p such that:

$$U=\{(t,t')\in V\times V'\},\quad \gamma\cap U=\{(t,t')\in V\times\{0\}\ :\ t>0\}$$

Again t is called a C^s (local) parameter for γ (centred on p), note that if p is a (C^0) endpoint of the 1-submanifold γ, then p is not a point of γ but is a limit point of γ.

If p is an enpoint of class C^1 for the C^1 1-submanifold γ parametrised by t, then we may define the outward tangent at p to be the vector $\frac{d\tilde{\gamma}}{dt}\in TM_p$. Note this is well-defined up to positive scalar multiples, and points along the submanifold away from p:

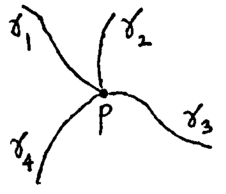
Meeting points:

By a meeting point of class C^s $(s=0,\ldots,r)$ of the C^r 1-submanifolds $\gamma_1,\ldots,\gamma_k$ of M we shall mean a point $p\in M$ such that there exists a neighbourhood of p and C^s coordinates (x,y) for $U\subset M$ centred on p such that:

$(\dagger)$ Here we use (t,t') to denote the point of U with coordinates (t,t').

$$\gamma_i \cap U = \{(x,y)\neq(0,0) \;;\; \arg(x+iy)=c_i\} \quad ,(i=1,\dots,k)$$

where $c_1,\dots,c_k$ are k distinct real constants. Thus in the (x,y) coordinates, $\gamma_1,\dots,\gamma_k$ are k half-lines meeting at the origin.



If M is a connected real-analytic surface, similar definitions can be given allowing differentiability classes C^r with $r=0,1,\dots,\infty,\omega$.

C Types of singularities - good singular points

Let $f:M\to N$ be a smooth mapping between smooth connected surfaces, and let $p\in M$. Two maps $f,f':M\to N$ have the same germ at $p\in M$ iff $f|_U = f'|_U$ for some sufficiently small neighbourhood U of p. We shall be interested in the possible germs of a mapping f at p, especially when p is a singular point,for this purpose, we may assume that f is defined on an arbitrarily small neighbourhood of p. For such a mapping f, we now distinguish between various types of singular points $p\in M$.

Choose local coordinates (x,y) on a neighbourhood U of p, (u,v) on a neighbourhood V of f(p), the the Jacobian of f is $J=u_x v_y - u_y v_x$. It has derivative $dJ=\frac{\partial J}{\partial x}dx + \frac{\partial J}{\partial y}dy$. Firstly p may be an ordinary point:

Proposition 2.1.2 (Canonical form for an ordinary point)

If p is an ordinary point, then there exist smooth coordinates (x,y), (u,v) centred on $p,f(p)$ such that f has the form:

$$u=x \ ,v=y \ .$$

Proof

By inversefunction theorem (standard).∎

Definition 2.1.3 (Whitney [W])

A singular point is called a good singular point if $dJ(p)\neq 0$.∎

Thus a good singular point is one which is not a critical point of the Jacobian.

Proposition 2.1.4 [W]

If p is a good singular point, $df(p)\neq 0$.

Proof

Direct calculation in local coordinates.∎

Now, by the implicit function theorem, if p is a good singular point, then there exists a neighbourhood U of p such that $\Sigma \cap U = \{p \in U : J(p)=0\}$ is a smooth connected 1-submanifold. Further, since $dJ(p) \neq 0$, by continuity of dJ, we can choose the neighbourhood U such that $\Sigma \cap U$ is a smooth connected 1-submanifold consisting of good singular points. We make the

Definition 2.1.5 [W]

A smooth connected 1-submanifold consisting of good singular points is called a general fold. ∎

The above remarks show:

Proposition 2.1.6

Any good singular point lies on a general fold, in fact we can choose a neighbourhood U of p such that $\Sigma \cap U$ is a general fold. ∎

Notes

(1) Any general fold is contained in a maximal general fold i.e. a maximal connected smooth 1-submanifold consisting of good singular points. We easily see that maximal general folds are precisely the connected components of the set of good singular points. Any good singular point lies on a unique maximal general fold.

(2) If $f:M \to N$ is a real-analytic map between connected real-analytic surfaces, then a general fold is a connected real-analytic 1-submanifold.

(3) Since $dJ(p) \neq 0$ at a good singular point, J has opposite signs on opposite sides of a general fold. ∎

D Order of a Good Singular Point

Let p be a good singular point for the mapping $f:M \to N$. Throughout this section σ will denote the general fold through p. Let σ have a local C^∞ parametrisation (§2.1B) $\tilde{\sigma}:V \to M$, $t \mapsto \tilde{\sigma}(t)$, where V is a neighbourhood of 0 in $\underline{\mathbb{R}}$. Choose smooth local coordinates on N defined in some neighbourhood of $f(p)$, and let f^1, f^2 denote (as usual) the components of f with respect to these coordinates.

Definition 2.1.7

The order of a good singular point p is the least positive integer r such that $\frac{d^r}{dt^r}(f^\gamma \circ \tilde{\sigma})(0) \neq 0$ for some $\gamma = 1,2$.

If no such r exists, i.e. if $\frac{d^r}{dt^r}(f^\gamma \circ \tilde{\sigma})(0) = 0 \ \forall r=1,2,\dots, \forall \gamma=1,2$, we say p has order ∞. ∎

It is easy to see that the definition is independent of the C^∞ parametrisation chosen for σ, and the local coordinates chosen for N.

Proposition 2.1.8

The order of p is the lesser degree of the lowest terms in the Taylor expansions of $f^1 \circ \tilde{\sigma}(t)$, $f^2 \circ \tilde{\sigma}(t)$ as power series in the parameter t.

Proof

Trivial. ∎

E Fold points

Defintion 2.1.9 [W]

A good singular point of order 1 is called a fold point. ∎

Thus a good singular point p is a fold point if and only if $\frac{d}{dt}(f^\gamma \circ \tilde{\sigma})(0) \neq 0$ for some $\gamma=1,2$. Alternatively, since $\frac{d}{dt}(f^1 \circ \tilde{\sigma})(0)$, $\frac{d}{dt}(f^2 \circ \tilde{\sigma})(0)$ are the components of the derivative $\frac{d}{dt}(f \circ \tilde{\sigma})(0) \in TN_{f(p)}$, the good singular point p is a fold point iff: $\frac{d}{dt}(f \circ \tilde{\sigma})(0) \neq 0$.

Alternatively, if $w = w_1 \frac{\partial}{\partial x} + w_2 \frac{\partial}{\partial y} \in TM_p$, the directional derivative $\nabla_w f(p)$ is the vector:

$$w_1 \frac{\partial f}{\partial x} + w_2 \frac{\partial f}{\partial y} \in TN_{f(p)}$$

Clearly (c.f.[W]) f is a fold point if and only if $\nabla_w f(p) \neq 0$ when w is tangent to σ at p.

Definition 2.1.10 [W]

A smooth connected 1-submanifold consisting of fold points is called a fold line. ∎

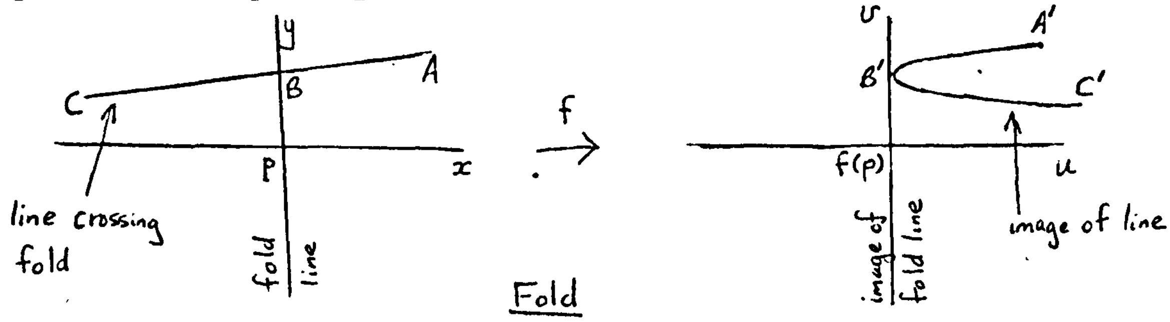
Thus a fold line is a special type of general fold.

Proposition 2.1.11 [W] (Canonical Form for Fold)

If p is a fold point, then there exist smooth coordinates (x,y) , (u,v) centred on p, f(p) respectively, such that f assumes the form:

$$u=x^2 \ , \quad v=y. \ \blacksquare$$

Here, the y-axis is the fold line, the mapping folding the coordinate neighbourhood of p along this line.



Fold

F Cusp points

Definition 2.1.12 [W]

A good singular point of order 2 is called a cusp point. ∎

Thus a good singular point is a cusp point if and only if

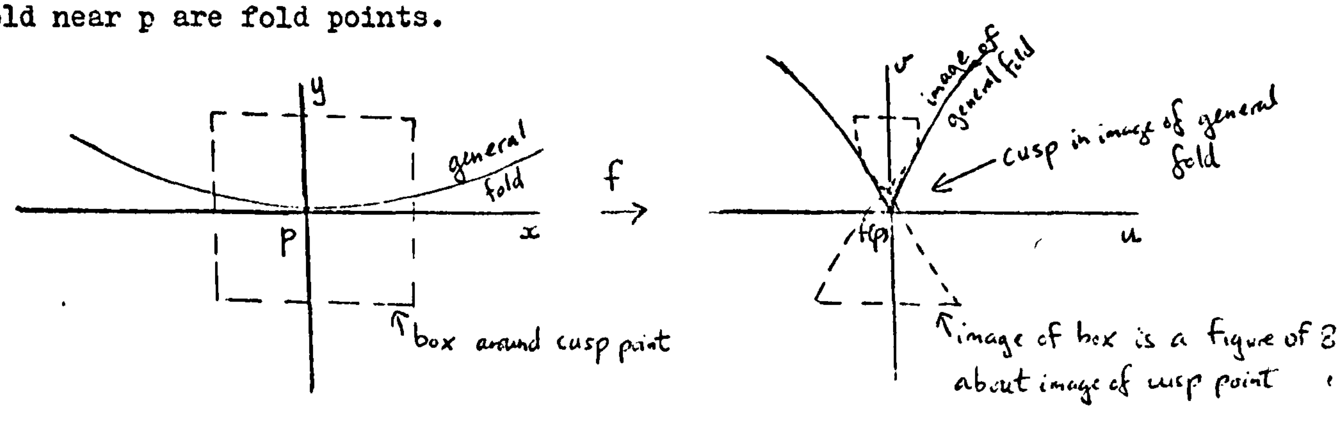
$$\frac{d}{dt}(f^{\gamma}\circ\tilde{\sigma})(0) = 0 \ \forall\gamma=1,2 \quad \text{and} \quad \frac{d^2}{dt^2}(f^{\gamma}\circ\tilde{\sigma})(0) \neq 0 \quad \text{for some } \gamma=1,2.$$

Proposition 2.1.13 [W]

If p is a cusp point, there exist smooth coordinates (x,y) , (u,v) centred on p, f(p) such that f has the form:

$$u = xy - x^3 \ , \quad v = y \ . \ \blacksquare$$

Thus p lies on the general fold $y=3x^2$. All other points on the general fold near p are fold points.



G Good Singular Points or order r , $2<r<\infty$

It appears not to be possible to give canonical forms for good singular points of order $r>2$, examples are provided by the formula:

$$u = xy - x^{r+1}\ ,\ v = y \quad (2<r<\infty)$$

which defines a C^∞ mapping with a good singular point of order r at (0,0).

Good singular points of order r , $2\leq r<\infty$ are isolated, in fact:

Proposition 2.1.14

If p is a good singular point of order r $(2\leq r<\infty)$, there exists a neighbourhood U of p such that all the points of $(\Sigma\cap U)\setminus p$ are fold points.

Proof

Suppose p is a good singular point of order r $(2\leq r<\infty)$. Then by (2.1.6), there exists a neighbourhood U" of p such that $\Sigma\cap U''$ is a general fold σ through p. Further, since a general fold is a smooth 1-submanifold, we can choose a neighbourhood $U'\subset U''$ of p, neighbourhoods V, V' of 0 in $\mathbb{R}$, and smooth coordinates (t,t') for U' centred on p such that:

$$U' = \{(t,t')\in V\times V'\}\ ,\quad \sigma\cap U' = \Sigma\cap U' = \{(t,t')\in V\times V' : t'=0\}$$

(c.f. § 2.1B). Parametrising the general fold $\sigma\cap U'$ by $\tilde{\sigma}: t\mapsto(t,0)$, since p is of order r:

$$\frac{d}{dt}(f^\gamma\circ\tilde{\sigma})(0) = \dots = \frac{d^{r-1}}{dt^{r-1}}(f^\gamma\circ\tilde{\sigma})(0) = 0 \quad \forall\gamma=1,2,\quad \frac{d^r}{dt^r}(f^\gamma\circ\tilde{\sigma})(0) \neq 0$$

for some $\gamma = 1,2$, say $\gamma=\gamma_0$. It follows [W p.392(b)] that there exist smooth functions T^γ defined on some neighbourhood W' of 0 such that:

$$\frac{d}{dt}(f^\gamma\circ\tilde{\sigma})(t) = t^{r-1}T^\gamma(t) \quad\text{and}\quad T^\gamma(0)\neq 0 \quad\text{for } \gamma=\gamma_0\ .$$

By continuity of T, it follows that:

$$\frac{d}{dt}(f^\gamma\circ\tilde{\sigma})(t)\neq 0 \quad\text{for } \gamma=\gamma_0 \quad\text{and } t\in\text{ some neighbourhood } W\subset W' \text{ of } 0,$$

$t\neq 0$.

Consider the set: $U = \{(t,t')\in(V\cap W)\times V'\}$. Then U is a neighbourhood of p and $(\Sigma\cap U)\setminus p$ consists of good singular points with $\frac{d}{dt}(f^\gamma\circ\tilde{\sigma})(t)\neq 0$ for some $\gamma = 1,2$, i.e. fold points. ∎

H Good Singular Points of order ∞

Recall that p is a good singular point of order ∞ if $\frac{d^r}{dt^r}(f^\gamma \circ \tilde{\sigma})(0)=0$ for all $r = 1,2,\ldots$, $\gamma=1,2$. An example is provided by the formula:

(2.1B) $u = xy - \exp(-x^{-2})$, $v = y$

This defines a C^∞ mapping with a good singular point of order ∞ at $(0,0)$.

I Collapse points

As usual, let p be a good singular point , and let σ be the general fold through p.

Definition 2.1.15

The good singular point p is called a collapse point if there exists a neighbourhood U of p such that $f|_{U\cap\sigma}$ = constant.

Definition 2.1.16

A smooth connected 1-submanifold consisting of collapse points is called a collapse line. ∎

We see that any collapse point is interior to a collapse line, and that a collapse line "collapses" to a point under the mapping.

Proposition 2.1.17

(a) A collapse point is a good singular point of order ∞.

(b) The converse is not necessarily true.

(c) If, however, the mapping $f:M\to N$ is a real-analytic mapping between real-analytic surfaces, then a good singular point $p\in M$ of order ∞ must be a collapse point.

Proof

(a) Let p be a good singular point lying on a general fold σ . If p is a collapse point, then there exists a neighbourhood U of p such that $f|_{U\cap\sigma}$= constant. Thus $f\circ\tilde{\sigma}$ is constant in a neighbourhood of t=0 (here $\tilde{\sigma}:t\mapsto\tilde{\sigma}(t)$ denotes a C^∞ parametrisation of σ, as usual. Hence $\frac{d^r}{dt^r}(f^\gamma\circ\tilde{\sigma})(0)=0$ $\forall\, r\geq1$, $\gamma=1,2$, thus p is a good singular point of order ∞.

(b) In the mapping (2.1B), the point (0,0) is a good singular point of order ∞, but is not a collapse point.

(c) Let $f:M\to N$ be C^{ω}, and let p be a good singular point of order ∞. As usual, p lies on a general fold σ, this is a real-analytic 1-submanifold of M, and thus has a C^{ω} parametrisation $\tilde{\sigma}:V\to\sigma\subset N$, $t\mapsto\tilde{\sigma}(t)$ where V is a neighbourhood of 0 in $\underline{R}$. Choosing C^{ω} coordinates for N, $f^{\gamma}\circ\tilde{\sigma}$ is analytic ($\gamma=1,2$). But by hypothesis: $\frac{d^r}{dt^r}(f^{\gamma}\circ\tilde{\sigma})(0)=0\ \forall r=1,2,\ldots,\ \forall\gamma=1,2$; therefore $f^{\gamma}\circ\tilde{\sigma}$ must be constant on V. It follows that p is a collapse point. ∎

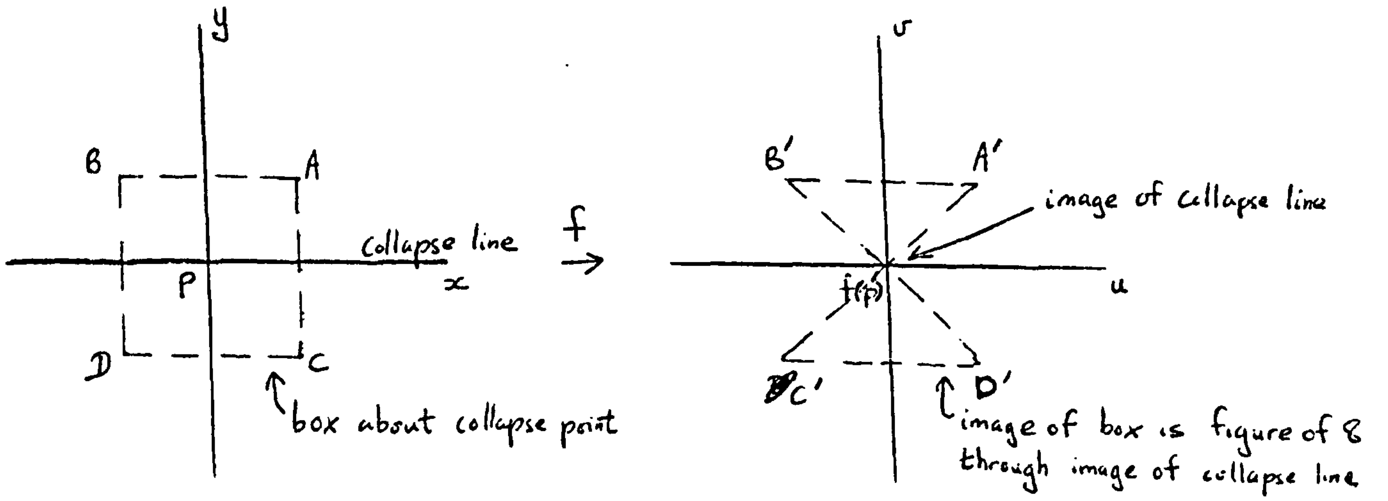
Proposition 2.1.18 (Canononical form for a collapse)

If p is a collapse point for the mapping $f:M\to N$, then there exist smooth coordinates (x,y) centred on p, (u,v) centred on f(p) such that f assumes the form:

$$u = xy \quad , \quad v = y \; .$$

Proof

See appendix 1. ∎



Collapse

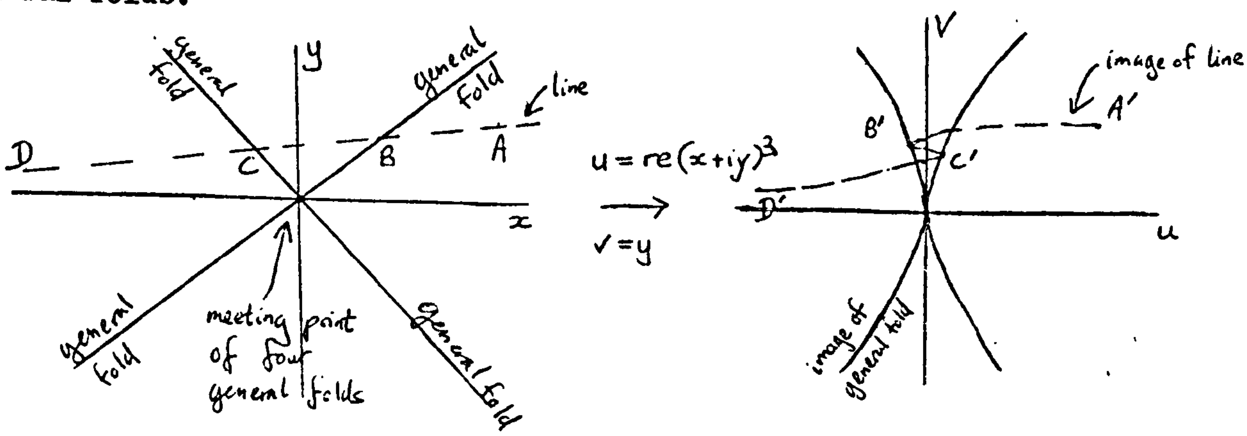
J Types of Singularities - Meeting Point of General Folds

We shall call p a C^s meeting point of general folds $\sigma_1, \dots, \sigma_k$ if $\sigma_1, \dots, \sigma_k$ are general folds, p is a C^s meeting point of the C^∞ 1-submanifolds $\sigma_1, \dots, \sigma_k$ as defined in §2.1A, and $\Sigma \cap U \setminus p = (\sigma_1 \cup \dots \cup \sigma_k) \cap U$ for some neighbourhood U of p.

As an example, consider the mapping defined by

$$u = \mathrm{re}(x+iy)^l \ , \ v = y$$

The Jacobian for this mapping = $\mathrm{re}\{l(x+iy)^{l-1}\}$, which is zero on the $2(l-1)$ half-lines $\arg(x+iy) = \pm\frac{\pi}{2(l-1)} + \frac{2r\pi}{l-1}$ $(r=0,1,\dots,l-1)$. These lines are easily seen to be general folds, thus (0,0) is a meeting point of 2(l-k) general folds.



Meeting point

K Types of singularities - branch points

Let $f:M\to N$ be a smooth map between connected smooth surfaces, and let $p\in M$.

Definition 2.1.19

We shall say f is ramified at p if there exists a neighbourhood U of p such that $f:U\setminus p\to N$ is a local homeomorphism (i.e. for all $p'\in U\setminus p$, there exists a neighbourhood U' of p', $U'\subset U$, such that $f|_{U'}$ is a homeomorphism onto its image). ∎

We see f is ramified at p iff either p is an isolated point of the branch set B of f or $p\notin B$.

By a Jordan region [A-S § 1.20] we shall mean an open set Δ of a surface whose closure can be mapped topologically onto a closed disc in such a way that Δ corresponds to the open disc.

Lemma 2.1.20

Let $f:M\to N$ be ramified at p. Then there exist neighbourhoods U^* of p, V^* of $f(p)$ such that:

(1) U^*, V^* are Jordan regions,

(2) the fundamental group of $U^*\setminus p$ is infinite cyclic,

(3) the fundamental group of $V^*\setminus f(p)$ is infinite cyclic,

(4) $f_*:\pi_1(U^*\setminus p)\to \pi_1(V^*\setminus f(p))$ is monomorphic.

Proof

Following Ahlfors & Sario [A-S § 1.20] if U is a connected locally compact Hausdorff topological space and N is a C^0 surface, a continuous mapping $f:U\to N$ is said to define U as a (ramified) covering surface of N if every $p'\in U$ has a neighbourhood U' such that $f:U'\setminus p'\to N$ is a local homeomorphism (*). Now let $f:M\to N$ be a smooth map between connected smooth surfaces, then it is clear that f is ramified at p if and only if there exists a neighbourhood U of p such that $f:U\to N$ defines U as a

(*) Ahlfors & Sario term a local homeomorphism a "smooth covering".

(ramified) covering surface of N. Our lemma now follows directly from Ahlfors & Sario's results for a covering surface, see [A-S §1.20D].

Let the generator of $\pi_1(U^*\setminus p)$ map to r times the generator of $\pi_1(V^*\setminus f(p))$. By 2.1.20 (4), $r\neq 0$, by suitable choice of generators we may assume $r>0$.

Definition 2.1.21

r is called the multiplicity of p (in the mapping f). By the phrase: "p is a point of multiplicity r" we shall mean f is ramified at p and p has multiplicity r. ∎

Definition 2.1.22

p is said to be a branch point of order (r-1) if f is ramified at p and p has multiplicity $r>1$. ∎

Proposition 2.1.23 (Canonical form for a branch point)

Let $f:M\to N$ be a smooth mapping between connected smooth surfaces. Let p be a point of multiplicity k. (Thus if $k>1$, p is a branch point of order k-1.) Then there exist C^0 coordinates (x,y), (u,v) centred on p, f(p) respectively such that f assumes the form:

$$w = z^k \quad \text{where } w \text{ denotes } u+iv,\ z \text{ denotes } x+iy\ .$$

Proof

If D denotes the disk $\{z\in\underline{C}\ ;\ |z|<1\}$ Ahlfors & Sario [A-S§1.20E] prove that there exist homeomorphisms $\phi:U\to D$, $\omega:V\to D$ of suitable neighbourhoods U,V of p,f(p) such that $\omega\cdot f(p') = \phi(p')^k$ for all $p'\in U$. Thus ϕ,ω define local coordinate charts, which define C^0 coordinates (x,y) for M, (u,v) for N, centred on p,f(p) respectively such that f assumes the form:

$$w = z^k \quad \text{where } w \text{ denotes } u+iv,\ z \text{ denotes } x+iy. \ ∎$$

Corollary 2.1.24

p is a point of multiplicity 1 if and only if there exist neighbourhoods U,V of p,f(p) respectively such that $f:U\to V$ is a homeomorphism.

Proof

Canonical form is now: $w=z$. ∎

Corollary 2.1.25

p is a branch point if and only if p is an isolated point of the branch set of f.

Proof

If p is an isolated point of the branch set B of f, it is clear that f is ramified at p (2.1.19). Further by (2.1.23) p cannot have multiplicity 1. Therefore by definition p is a branch point. Conversely, if p is a branch point, then by (2.1.19) either p is an isolated point of the branch set or $p \notin B$. In the latter case (2.1.24) shows that p has multiplicity 1 and therefore is not a branch point. ∎

Remarks 2.1.26

Thus we have been discussing isolated points of the branch set; note if p is an isolated singular point then according to (2.1.19) f is ramified at p and thus, by (2.1.22), (2.1.24), either p is a branch point of f or f is a local homeomorphism at p. An example of the latter case is provided by the formula :

$$u = \frac{1}{3}x^3 + xy^2 \quad , \quad v=y$$

which defines a homeomorphism (of the plane to the plane) with an isolated singularity at (0,0). (0,0) is, of course, a point of multiplicity 1.

2.2 The singularities of a harmonic map at a point p

A Expressions and inequalities for df, J, dJ

Let M,N be Riemann surfaces and let $f:M\to N$ be harmonic with respect to some hermitian metric h on N. Let $p\in M$. We shall classify into four basic types the singularities possible at p. For this purpose it suffices that f be defined in an arbitrarily small neighbourhood of p, i.e. we are interested in the germ of f at p.

Our starting point is theorem (1.4.8). Choose smooth coordinates (x,y) , (u,v) centred on p, f(p) respectively such that:

$$(2.2A)\begin{cases}(1)\ (x,y)\text{ are isothermal at p }(1.2.3),\\(2)\ (u,v)\text{ are normal coordinates centred on }f(p)\ ;\end{cases}$$

suppose these coordinates are admissible (see (1.4.8)). Then in some neighbourhood U of p, f assumes one of the forms:

$$(2.2B)\begin{cases}(a)\ u=0\ ,\ v=0\\(b)\ u=0\ ,\ v=re(b_kz^k)+o(\rho)^k\\(c)\ u=re(a_lz^l)+o(\rho)^l\ ,\ v=re(b_kz^k)+o(\rho)^k\end{cases}$$

where l,k are positive integers, $z=x+iy$, $\rho=|z|$, a_l, b_k are non-zero complex numbers and in case (c):

(2.2C) either $l>k$ or $l=k$ and $im(a_l\bar{b}_k)\neq 0$.

According to (1.4.8) admissible coordinates can be obtained from any coordinates satisfying (2.2A) by a linear change of (u,v) coordinates.

We shall classify the possible germs of f at p according to the values of l,k; note that if we set $l=k=\infty$ in case (a), $l=\infty$ in case (b), it can be shown that (l,k) are invariant under different choice of admissible coordinates (see remarks in §1.4). By (1.4.8) the derivative and Jacobian in the three cases (2.2B) (a),(b),(c) are given in the neighbourhood U by:

(2.2D)

(a) $df \equiv 0$, $J \equiv 0$

(b) $df = \begin{bmatrix} 0 & 0 \\ re(kb_k z^{k-1}) + o(\rho)^{k-1} & -im(kb_k z^{k-1}) + o(\rho)^{k-1} \end{bmatrix}$

$J \equiv 0$

(c) $df = \begin{bmatrix} re(la_1 z^{l-1}) + o(\rho)^{l-1} & -im(la_1 z^{l-1}) + o(\rho)^{l-1} \\ re(kb_k z^{k-1}) + o(\rho)^{k-1} & -im(kb_k z^{k-1}) + o(\rho)^{k-1} \end{bmatrix}$

$J = lk|z|^{2k-2}.im(a_1\bar{b}_k z^{l-k}) + o(\rho)^{l+k-2}$

Now define $|df|$ at the point $(x,y) \in U$ by:

$$|df|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

Note this is not independent of the coordinates employed. (C.f. $\|df\|$ in §1.1D)

Note $|df| = 0$ if and only if $df=0$.

Lemma 2.2.1 (Estimate for df)

if, in (2.2B), $k<\infty$, i.e. we have case (b) or (c), then there exists a neighbourhood U' of p and a constant $A>0$ such that:

$|df| \geqslant A|z|^{k-1}$ at all points (x,y) of U' (where z=x+iy as usual).

Proof

In case (2.2B)(b), df is given by (2.2D)(b) and hence:

$$|df|^2 = \{re(kb_k z^{k-1})\}^2 + \{im(kb_k z^{k-1})\}^2 + o(\rho)^{2k-2}$$

Thus

$$|df|^2 = |kb_k z^{k-1}|^2 + o(\rho)^{2k-2}$$

In case (2.2B)(c), df is given by (2.2D)(c) and hence:

$$|df|^2 = \{re(la_1 z^{l-1})\}^2 + \{im(la_1 z^{l-1})\}^2 + o(\rho)^{2l-2}$$
$$+ \{re(kb_k z^{k-1})\}^2 + \{im(kb_k z^{k-1})\}^2 + o(\rho)^{2k-2}$$

In either case (b) or (c), since $l \geqslant k$, we get:

$$|df|^2 = 2A^2|z|^{2k-2} + o(\rho)^{2k-2}$$

where $2A^2$ is a constant.

Now choose a neighbourhood U' of p such that:

$$\left| o(\rho)^{2k-2} \right| < A^2 |z|^{2k-2}$$

then it follows that:

$$|df|^2 > A^2 |z|^{2k-2}$$

and the desired inequality follows. ∎

Corollary 2.2.2 (df has isolated zeros)

Provided df is not identically zero on a neighbourhood of p, then there exists a neighbourhood U' of p such that $df(q) \neq 0$ for all $q \in U' \setminus \{p\}$.

Proof

Immediate from (2.2.1). ∎

Similarly we define $|dJ|$ by

$$|dJ|^2 = \left|\frac{\partial J}{\partial x}\right|^2 + \left|\frac{\partial J}{\partial y}\right|^2 \quad ;$$

a straitforward calculation from (2.2D)(c) gives the following expression for dJ:

(2.2E)
$$\frac{\partial J}{\partial x} + i\frac{\partial J}{\partial y} = lkz\,|z|^{2k-4}\left\{ (l+k-2)\,\mathrm{im}(a_l \bar{b}_k z^{l-k}) + i(l-k)\,\mathrm{re}(a_l \bar{b}_k z^{l-k}) \right\} + o(\rho)^{l+k-3}$$

$$(1 < k \leq l < \infty)$$

(2.2F)
$$\frac{\partial J}{\partial x} + i\frac{\partial J}{\partial y} = l(l-1)\left\{ \mathrm{im}(a_l \bar{b}_k z^{l-2}) + i\,\mathrm{re}(a_l \bar{b}_k z^{l-2}) \right\} + o(\rho)^{l-2}$$

$$(1 = k < l < \infty)$$

Lemma 2.2.3 (Estimate for dJ)

If in (2.2B), $l,k < \infty$, i.e. we have (2.2B)(c) and $(l,k) \neq (1,1)$, then there exists a neighbourhood U of p and a constant $B > 0$ such that

$$|dJ| \geqslant B|z|^{l+k-3}$$ for all points (x,y) of U' (where z=x+iy as usual).

Proof

From (2.2E), if $k > 1$, noting that the hypothesis $(l,k) \neq (1,1)$ implies $l+k \geqslant 3$:

$$|dJ|^2 = l^2k^2\left\{|z||z|^{2k-4}\right\}^2\left\{(l+k-2)^2\left\{\mathrm{im}(a_l \bar{b}_k z^{l-k})\right\}^2 + (l-k)^2\left\{\mathrm{re}(a_l \bar{b}_k z^{l-k})\right\}^2\right\} + o(\rho)^{2l+2k-6}$$

Hence, since $l+k-2 \geqslant 1, l \geqslant 1, k \geqslant 1$,

$$|dJ|^2 \geqslant \left\{|z|^{2k-3}\right\}^2 |a_l \bar{b}_k|^2 |z^{l-k}|^2 + o(\rho)^{2l+2k-6} \qquad \text{(provided } l \neq k)(^*)$$

Thus,

$$|dJ|^2 \geqslant |a_l \bar{b}_k|^2 |z|^{2l+2k-6} + o(\rho)^{2l+2k-6} \qquad \text{(provided } l \neq k)(^*)$$

If k=1, we get a similar result from (2.2?).

The desired inequality now follows in a similar fashion to that in (2.2.1). ∎

Corollary 2.2.4 (Bad singularities isolated)

Provided J is not identically zero on a neighbourhood of p, then there exists a neighbourhood U of p such that all the singular points in U∖p are good singular points.

Proof

If in (2.2B) we have (a) or (b), J is identically zero on a neighbourhood of p; thus we consider case (c), $l,k<\infty$.

If now $(l,k)=(1,1)$, (2.2D)(c) shows that $J= \mathrm{im}(a_1\bar{b}_1) + o(\rho)^0$ where (by (2.2C)) $\mathrm{im}(a_1\bar{b}_1) \neq 0$. It follows that J is non-zero on some neighbourhood of p, i.e. there exists a neighbourhood U of p such that there are no singular points in U.

Alternatively, if $(l,k) \neq (1,1)$, (2.2.3) shows that $|dJ(q)| > 0$ for all q in a neighbourhood U of p, save possibly for q=p.

The corollary now follows. ∎

(*) If l=k, these expressions need slight modifications.

B The Singularities for different values of (l,k)

Let $f:M\to N$ be a map between Riemann surfaces which is harmonic with respect to a hermitian metric h on N. Let $p\in M$. Let (x,y) , (u,v) be admissible coordinates centred on p , $f(p)$. Then f assumes one of the forms (2.2B) (a),(b) or (c) , with condition (2.2C) satisfied in case (c). We now classify the different possible singularities at p according to the values of l,k in the expression (2.2B) for f. (†)

Proposition 2.2.5

If, in (2.2B), $l=k=\infty$, there exists a neighbourhood U of p such that $f(U)$ = point. Then $\Sigma\cap U = \Sigma_2\cap U = U$. (see further (3.1.1))

Proof

In (2.2B) we have case (a), thus $u\equiv 0$, $v\equiv 0$ in a neighbourhood U of p, thus from (2.2D)(a), $J\equiv 0$, $df\equiv 0$ in U. Hence $f(U)$ = point, and $\Sigma\cap U = \Sigma_2\cap U = U$. ∎

Proposition 2.2.6

If, in (2.2B), $l=\infty$,$k<\infty$, there exists a neighbourhood U of p such that $f(U)$ is a geodesic arc γ . If s denotes arc lengh along γ , s is a harmonic function on U. Further $\Sigma\cap U = U$, and we can choose U such that:

if $k=1$, $f:U\to\gamma$ has no critical points (§2.1A). Then $\Sigma_2\cap U = \emptyset$;

if $k>1$, $f:U\to\gamma$ has critical point only at p. Then $\Sigma_2\cap U = \{p\}$.

Proof

In (2.2B) we have case (b), thus $u\equiv 0$, $v\not\equiv 0$ on a neighbourhood U of p, thus f maps U into the axis γ: $u=0$. This is a geodesic arc (1.4.7)(a). Now from the harmonic equations (1.1G) (with $\gamma=2$) we get:

$$(2.2G)\qquad \Delta f^2 + g^{ij}f_i^{\alpha}f_j^{\beta}L_{\alpha\beta}^{\ 2} = 0$$

and since $f^2\equiv v$, $f^1\equiv u\equiv 0$, and by (1.4.7), $L_{22}^{\ 2}(0,v) = 0$ $\forall\, v$, (2.2G) reduces to:

$$\Delta v = 0.$$

But, if s denotes arc length along γ , by definition of normal coordinates,

(†) As usual, we set $l=k=\infty$ in case (a) of (2.2B), $l=\infty$ in case (b).

v = constant multiple of s, therefore,

$$\Delta s = 0 \ .$$

Thus arc length along γ is a harmonic function on U.

Now from (2.2D)(b), if $k=1$, $df(p) \neq 0$, whereas if $k>1$, $df(p) = 0$. In either case, by (2.2.2) there exists a neighbourhood U' of p such that $df(q) \neq 0$ for $q \in U' \setminus p$. The proposition follows. ∎

Proposition 2.2.7

If, in (2.2B), $l=k=1$, p is an ordinary point. There exists a neighbourhood U of p such that $\Sigma \cap U = \emptyset$. Further we can choose U such that $f|_U$ is a smooth diffeomorhpism, and such that in suitable smooth coordinates (which will not, in general, be admissible coordinates) (x',y') , (u',v') centred on p, f(p) , f assumes the form:

$$u' = x' \ , \ v' = y' \ .$$

Proof

In (2.2B), we have case (c) with $l=k=1$. From (2.2D)(c) we get $J = im(a_1\bar{b}_1) + o(\rho)^0$. By (2.2C), $im(a_1\bar{b}_1) \neq 0$. Thus $J(p) \neq 0$. Thus there exists a neighbourhood U of p such that $J(q) \neq 0$ for all $q \in U$. The rest of the proposition follows from the inverse function theorem. ∎

Proposition 2.2.8

(a) If, in (2.2B), $l=2$, $k=1$, p is a good singular point (2.1.3) and lies on a general fold σ (2.1.5); in fact, there exists a neighbourhood U of p such that $\Sigma \cap U$ is a general fold. J has opposite signs on opposite sides of this general fold, and $\Sigma_2 \cap U = \emptyset$. p may be a fold point, in which case U can be chosen such that in suitable smooth coordinates (x',y'), (u',v') centred on p, f(p) , f assumes the form:

(2.2H) $\quad u' = x'^2 \ , \ v' = y'$ on U,

or p may be a cusp point, in which case, in suitable smooth coordinates f assumes the form:

$$u' = x'y' - x'^3 \ , \ v' = y' \ ,$$

or p may be a good singular point of order > 2 (2.1.7) - as a special case p may be a collapse point (2.1.15), in which case, in suitable smooth coordinates, f assumes the form:

$$u = x'y', v' = y'.$$

(b) In the case that p is a fold point, then p lies on a fold line σ; if this has image $f(\sigma)$ with non-zero geodesic curvature at $f(p)$, we can find a neighbourhood U of p such that $f(U)$ lies entirely on the convex side of $f(\sigma)$.

Proof

(a) In (2.2B) we have case (c) with $l=2$, $k=1$; from (2.2D)(c), $J = \mathrm{im}(a_2\bar{b}_1 z) + o(\rho)^1$. Thus $J(p) = 0$. Differentiating, $dJ(p)$ has components:

$$\frac{\partial J}{\partial x} = \mathrm{im}(a_2\bar{b}_1) + o(\rho)^0, \quad \frac{\partial J}{\partial y} = \mathrm{re}(a_2\bar{b}_1) + o(\rho)^0,$$

and since $a_2 \neq 0$, $b_1 \neq 0$ so that $a_2\bar{b}_1 \neq 0$, then $dJ(p) \neq 0$. Thus p is a good singular point and thus, by (2.1.6), p lies on a general fold - in fact there exists a neighbourhood U of p such that $\Sigma \cap U$ is a general fold. By (2.1.4), $\Sigma_2 \cap U = \emptyset$. Now let r = order of the good singular point p (2.1.7), then if $r = 1$, p is a fold point (2.1.9), if $r = 2$, p is a cusp point (2.1.12), otherwise $r > 2$; if $r = \infty$, p may be a collapse point (2.1.15). Canonical forms are given by (2.1.11), (2.1.13) and (2.1.18).

(b) If p is a fold point, then in suitable coordinates, f assumes the form (2.2H) where $\sigma : x' = 0$ is the fold line and $f(\sigma): u' = 0$ is its image. Since $u' = x'^2 \geq 0$ for all points $(x', y') \in U$, it is clear that we can find a neighbourhood U such that $f(U)$ lies entirely on one side of $f(\sigma)$. If $f(\sigma)$ has non-zero geodesic curvature, by Sampson's maximum principle (1.2.12) this cannot be the concave side. Thus $f(U)$ lies entirely on the convex side of $f(\sigma)$. ∎

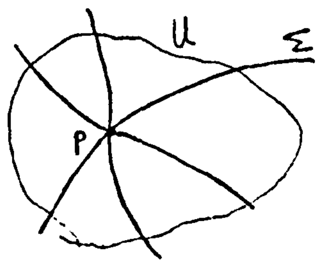
Proposition 2.2.9

If, in (2.2B), $l > k$ and $(l,k) \neq (2,1)$, then p is not a good singular point. It is a C^1 meeting point of $2(l-k)$ general folds (§2.2J). Such points are isolated. The angle (1.3.1) between the outward tangents at p along any two adjacent folds is $\pi/(1-k)$. Thus the general folds are arranged at equal angles around p. J has opposite signs on opposite sides of each general fold.

There exists a neighbourhood U of p such that $\Sigma \cap U = p \cup 2(1-k)$ disjoint smooth 1-submanifolds with p a C^1 endpoint (§2.1B') of each. Further we can choose U such that:

if $k=1$, $\Sigma_2 \cap U = \emptyset$,

if $k>1$, $\Sigma_2 \cap U = \{p\}$.



$1-k = 3$

p is meeting point of 6 general folds. Angle between adjacent general folds $= \pi/3$

Example of meeting point - sketch of $\Sigma \cap U$

Proof

In (2.2B) we have case (c) with $1>k$ and $(1,k) \neq (2,1)$. Note the Jacobian is given by (2.2D)(c) viz: $J = 1k|z|^{2k-2}.\mathrm{im}(a_1\bar{b}_k z^{1-k}) + o(\rho)^{1+k-2}$ thus $J(p) = 0$. From (2.2E), if $k>1$, or (2.2F) if $k=1, 1>2$, we see that $dJ(p) = 0$. Thus p is a singular point but not a good singular point. Also p is a critical point for the Jacobian J. By (2.2.3) there exists a neighbourhood U of p and a constant $B>0$ such that

(2.2I) $\quad |dJ| \geq B|z|^{1+k-3}$ throughout U.

Note, in the case at hand, $1+k-3 \geq 1$.

We now apply results of Kuiper [Kp] on C^1- equivalence of functions near isolated critical points. We consider the Jacobian J as a function of the coordinates (x,y) on some neighbourhood of 0 in $\mathbb{R}^2$. Then J has an isolated critical point (0,0) and in Kuiper's terminology, (2.2I) tells us that "J has property $\Omega(1+k-2)$ " [Kp p.200]. By [Kp theorem A p.208 and p.212], there exists a C^1 change of coordinates $h: W \to W' = h(W)$, $(x,y) \mapsto (x',y')$ with $h(0,0) = (0,0)$ (W, W' neighbourhoods of $(0,0)$ in $\underline{R}^2$) such that:

(a) in the new coordinates, J is given in some neighbourhood of p by:

$$(2.2J) \qquad J = 1k|z'|^{2k-2} \cdot \operatorname{im}(a_1\overline{b}_k z'^{1-k})$$

where $z' = x' + iy'$, (c.f. (2.2D)(c) - we have removed the remainder term $o(\rho)^{1+k-2}$),

(b) $h: W \to W'$ is C^∞ on $W \setminus (0,0)$ and $dh(0,0) = \text{identity}$.

Note the new coordinates $(x'y')$ are C^1 coordinates centred on p; also, since $dh(0,0) = \text{identity}$, and the old coordinates (x,y) were isothermal at p, it follows that the new coordinates are isothermal at p.

Now, in some neighbourhood U of p, J is given by (2.2J) above, therefore:

$$J=0 \Leftrightarrow \operatorname{im}(a_1\overline{b}_k z'^{1-k}) = 0 \ (\text{or } z'=0) \Leftrightarrow \arg(a_1\overline{b}_k z'^{1-k}) = 0 \text{ or } \pi \ (\text{mod } 2\pi)$$

$$(1-k)\arg z' = -\arg(a_1\overline{b}_k) + r\pi \quad (r=0,\pm 1,\pm 2,\ldots)$$

$$\arg z' = -\frac{1}{1-k}\arg(a_1\overline{b}_k) + \frac{r\pi}{1-k} \quad (r=0,\pm 1,\pm 2,\ldots)$$

We get $2(1-k)$ distinct values for $\arg z'$ by taking $r=0,1,\ldots,2(1-k)-1$.

Thus $\Sigma \cap U \setminus p = \bigcup_{r=1}^{2(1-k)-1} (\gamma_r \cap U)$ where γ_r is the smooth 1-submanifold given in the C^1 coordinates (x',y') by:

$$(2.2K) \qquad \arg z' = -\frac{1}{1-k}\arg(a_1\overline{b}_k) + \frac{r\pi}{1-k}$$

Referring to the definition (§2.1B), p is thus a C^1 meeting point of the $2(1-k)$ disjoint smooth 1-submanifolds γ_r.

Further, by (2.2.3), we can choose the neighbourhood U such that $dJ(q) \neq 0$ for all $q \in U \setminus p$. Then each γ_r must be a general fold. Thus p is a C^1 meeting point of general folds (§2.1K), and

$$\Sigma \cap U = p \cup \text{the } 2(1-k) \text{ disjoint general folds } \gamma_r.$$

We now show that the angle (1.3.1) between the outward tangents at p along adjacent folds is $\pi/(1-k)$. For the outward tangents are respectively:

$$t = t^1 \frac{\partial}{\partial x'} + t^2 \frac{\partial}{\partial y'} \quad , \quad t' = t'^1 \frac{\partial}{\partial x'} + t'^2 \frac{\partial}{\partial y'} \quad ,$$

then the angle between them is given (1.3.4) by:

$$\cos^{-1} \frac{t^1 t'^1 + t^2 t'^2}{\sqrt{\{(t^1)^2 + (t^2)^2\}}\sqrt{\{(t'^1)^2 + t'^2)^2\}}}$$

It is clear that this is precisely the same as the formula for the angle between two adjacent lines (2.2K) considered as lines in $\underline{C} = \underline{R}^2$, i.e. $\pi/(1-k)$. Thus the general folds are arranged at equal angles of $\pi/(1-k)$ around p.

We also describe $\Sigma_2 \cap U$. By (2.2D)(c), if k =1, then df(p) $\neq 0$; if $k > 1$ so that $l > 1$, then $df(p) \not\equiv 0$; further, in either case, since $(\Sigma \cap U) \setminus p$ consists of good singular points, then by (2.1.4), $df(q) \neq 0 \ \ \forall q \in (\Sigma \cap U) \setminus p$.

Therefore: $\Sigma_2 \cap U = \begin{cases} \emptyset & \text{if } k=1 \\ \{p\} & \text{if } k>1. \end{cases}$

Lastly, it is clear that a meeting point of general folds is isolated. ∎

<u>Proposition 2.2.10</u>

If, in (2.2B), l=k >1, then p is a branch point of order k-1 (§2.1D). There exists a neighbourhood U of p such that $\Sigma \cap U = \Sigma_2 \cap U = \{p\}$. Further we can choose U such that in suitable C^0 coordinates (x',y'), (u',v') centred on p, f(p) respectively, f assumes the form:

$$w' = z'^k \ , \text{ where } \ w'=u'+iv', \ z'=x'+iy' \ , \ (x',y') \in U.$$

Thus each point of f(U) has k preimages in U.

Branch points are isolated.

<u>Proof</u>

From (2.2B)(c) with $l=k>1$,

$$J = k^2 |z|^{2k-2} . \operatorname{im}(a_1 \bar{b}_k) + o(\rho)^{2k-2}$$

Thus $J(p)=0$ and there exists a neighbourhood U of p such that $J(q) \neq 0$ $\forall q \in U \setminus p$. Thus by the inverse function theorem, $f: U \setminus p \to N$ is a local homeomorphism (in fact a local diffeomorphism) and hence, by definition, f is ramified at p (see (2.1.19)). Note also that (2.2D)(c) shows that $df(p)=0$ and thus $\Sigma \cap U = \Sigma_2 \cap U = \{p\}$.

We now show that p has multiplicity k.

Firstly, from (2.2B)(c), with respect to our admissible coordinates $(x,y),(u,v)$, f takes the form:

$$u = \mathrm{re}(a_k z^k) + o(\rho)^k \quad , \quad v = \mathrm{re}(b_k z^k) + o(\rho)^k$$

This can be written:

$$u = \mathrm{re}\, a_k \,\mathrm{re}\, z^k - \mathrm{im}\, a_k \,\mathrm{im}\, z^k + o(\rho)^k$$

$$v = \mathrm{re}\, b_k \,\mathrm{re}\, z^k - \mathrm{im}\, b_k \,\mathrm{im}\, z^k + o(\rho)^k$$

By (2.2C), $\mathrm{im}(a_k \bar{b}_k) \neq 0$, therefore

$$u' = \lambda\{(\mathrm{im}\, b_k)u - (\mathrm{im}\, a_k)v\} \quad , \quad v' = \lambda\{(\mathrm{re}\, b_k)u - (\mathrm{re}\, a_k)v\}$$

where $\lambda = 1/(\mathrm{im}\, b_k \,\mathrm{re}\, a_k - \mathrm{im}\, a_k \,\mathrm{re}\, b_k)$ defines a non-singular linear change of coordinates; in the new coordinates, on dropping dashes f takes the form:

$$u = \mathrm{re}(z^k) + o(\rho)^k \quad , \quad v = \mathrm{im}(z^k) + o(\rho)^k \ ,$$

or in complex notation, $z=x+iy$, $w=u+iv$,

(2.2L) $\quad w = z^k + o(\rho)^k$

Now, by (2.1.20), we can choose neighbourhoods U^* of p, V^* of $f(p)$ such that U^*, V^* are Jordan regions with $\pi_1(U^* \setminus p)$, $\pi_1(V^* \setminus f(p))$ both infinite cyclic groups. Consider $f_* : \pi_1(U^* \setminus p) \to \pi_1(V^* \setminus f(p))$. It maps a generator of $\pi_1(U^* \setminus p)$ to a multiple of a generator of $\pi_1(V^* \setminus f(p))$, we shall show that this multiple is k.

For the generator of $\pi_1(U^* \setminus p)$ is given by the homotopy class of the mapping $s: [0,1] \to U^* \setminus p$, $t \mapsto z = \varepsilon.\exp(2\pi i t)$ where $\varepsilon > 0$ is chosen so small that the image of s, namely the circle centre p radius ε, lies within $U^* \cap f^{-1}(V^*)$. Consider the composition:

$$[0,1] \xrightarrow{s} U^* \setminus p \xrightarrow{f} V^* \setminus f(p)$$

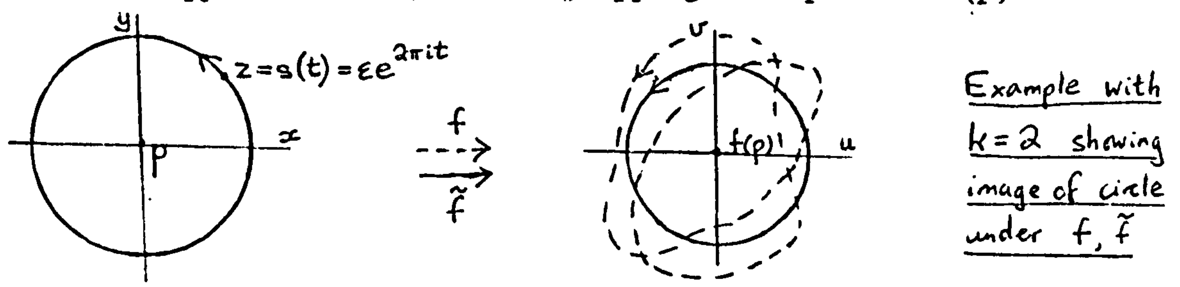
Claim: $f \circ s$ has homotopy class k times the generator of $\pi_1(V^* \setminus f(p))$, provided $\varepsilon > 0$ is chosen small enough.

Proof of claim: First replace f by the auxiliary mapping $\tilde{f}$ given in our local coordinates by:

$$w = z^k .$$

Then $\tilde{f} \circ s : [0,1] \to V^* \setminus f(p)$ is given by $t \mapsto \varepsilon^k(\exp(2\pi i t))^k = \varepsilon^k \exp(2\pi i k t)$. As t increases from 0 to 1, $\tilde{f} \circ s(t)$ travels round the circle of radius ε^k k times, thus certainly $\tilde{f} \circ s$ has homotopy class k times the generator.

Now choose $\varepsilon > 0$ so small that the term $o(\rho)^k$ in (2.2L) is less than $\frac{1}{2}|z|^k$ in absolute value. Then the mapping $s \mapsto z^k + s.o(\rho)^k$, $s \in [0,1]$ defines a homotopy between $\tilde{f}$ and f as mappings $U^* \setminus p \to V^* \setminus f(p)$.



Thus $\tilde{f} \circ s$ and $f \circ s$ are homotopic mappings: $[0,1] \to V^* \setminus f(p)$ and hence $f \circ s$ also has homotopy class k times the generator of $\pi_1(V^* \setminus f(p))$.

(Basically we have shown that if a point q encircles p once in M, f(q) encircles f(p) k times in N.)

Thus $f_* : \pi_1(U^* \setminus p) \to \pi_1(V^* \setminus f(p))$ maps the generator of $\pi_1(U^* \setminus p)$ to k times the generator of $\pi_1(V^* \setminus f(p))$, therefore p has multiplicity k, and since, by hypothesis, $k > 1$, p is a branch point of order k-1.

The canonical form $w' = z'^k$ follows from (2.1.22).

Note, lastly, by (2.1.25) branch points are isolated. ∎

C Summary of Possible Singularities and Main Theorem

We summarise all the possible singularities for the various cases of (l,k) in the following table by collecting together information from propositions (2.2.5) to (2.2.10).

Propn	Case	$\Sigma \cap U$	$\Sigma_2 \cap U$	behaviour of f in a neighbourhood of p
2.2.5	$l=\infty, k=\infty$	U	U	f maps a neighbourhood of p to a point
2.2.6	$l=\infty, 1<k<\infty$	U	p	f maps a neighbourhood U' of p into a geodesic arc, $f:U' \to \gamma$ has critical point at p
2.2.6	$l=\infty, k=1$	U	ϕ	same, but no critical point at p
2.2.7	$l=k=1$	ϕ	ϕ	p is an ordinary point
2.2.8	$l=2, k=1$	general fold	ϕ	p lies on general fold
2.2.9	$l>2, k=1$	$p \cup 2(l-k)$ disjoint general folds	ϕ	p is a C^1 meeting point of 2(l-k) general folds
2.2.9	$l>k, k>1$	ditto	p	ditto
2.2.10	$l=k>1$	p	p	p is a branch point of order k-1

TABLE (2.2M)

Intepretation of table

Example: last line is to be read: "if $l=k>1$, there exists a neighbourhood U of p such that $\Sigma \cap U = \{p\}$, $\Sigma_2 \cap U = \{p\}$; p is a branch point of order k-1; for further information see proposition (2.2.10)".

We can now state our main theorem of this chapter.

Theorem 2.2.11 (Classification of Singularities of a Harmonic Map)

Let M,N be Riemann surfaces and let $f:M\to N$ be harmonic with respect to some hermitian metric on N. Let Σ denote the set of singular points of M and let Σ_2 denote the subset on which the derivative df vanishes. Let $p\in\Sigma$. Then there exists a neighbourhood U of p such that one of the following holds:

(1) $\Sigma\cap U = U$. Then U can be chosen so that either (a) f is constant on U, then $\Sigma_2\cap U = U$, or (b) $f(U)$ is a geodesic arc; arc length along the geodesic is a harmonic function on the neighbourhood U; U may further be chosen so that $\Sigma_2\cap U = \{p\}$ or ϕ. (*)

(2) $\Sigma\cap U$ is a general fold. Then $\Sigma_2\cap U = \phi$. U can further be chosen such that either (a) $\Sigma\cap U$ is a fold line or a collapse line, or (b) p is a good singular point of order r $(2\leq r\leq\infty)$. If $\Sigma\cap U$ is a fold line, then if $f(\Sigma\cap U)$ has non-zero geodesic curvature at $f(p)$, U may further be chosen such that f maps U to the convex side of the image $f(\Sigma\cap U)$ of the fold line.

(3) $\Sigma\cap U = \{p\}\cup$ even number of disjoint general folds with p an endpoint of each. Then p is a C^1 meeting point of these general folds, and the general folds are arranged at equal angles around p (see (2.2.9). $\Sigma_2\cap U = \{p\}$ or ϕ. Such points p are isolated.

(4) $\Sigma\cap U = \{p\}$. Then $\Sigma_2\cap U = \{p\}$ and p is a branch point. Such points are isolated.

Note

Definition of terms are to be found as follows:

general fold: (2.1.5), fold line: (2.1.10), collapse line: (2.1.16), good singular point of order r : (§2.1G), endpoint: (§2.1B), C^1 meeting point: (§2.1J), branch point: (2.1K)

(*) We shall see in §3.1 that (provided M,N are connected) in case (1)(a) f is constant on the whole of M and in case (1)(b) f maps the whole of M to a geodesic arc.

Proof of theorem 2.2.11.

Choose admissible coordinates (x,y) , (u,v) centred on p, f(p), then f is given by one of the expressions (2.2B) for some value of (l,k). The possibilities corresponding to the different values of (l,k) are listed in propositions (2.2.5) to (2.2.10) and summarised in table (2.2M) above.

(1) If $\Sigma \cap U = U$, we must have case $(l=\infty, k=\infty)$ or case $(l=\infty, k<\infty)$. We thus apply proposition (2.2.5) or (2.2.6).

(2) If $\Sigma \cap U$ is a general fold, we must have case $(l=2, k=1)$. We thus apply proposition (2.2.8).

(3) If $\Sigma \cap U = \{p\} \cup$ even number of disjoint general folds with p an endpoint of each, we must have case ($l>k$ and $(l,k) \neq (2,1)$). We thus apply proposition (2.2.9).

(4) If $\Sigma \cap U = \{p\}$, we must have case $(l=k>1)$. We thus apply proposition (2.2.10).

The only possibility ommitted is the case $((l,k)=(1,1))$, but by proposition (2.2.7), p is then an ordinary point contrary to hypothesis.

The theorem is thus proven. ∎

2.3 Realisation of each Singularity

Theorem 2.3.1

Each type of singularity (2.2.11) (1),(2),(3),(4) can be realised at the origin for a harmonic map $f:\mathbb{R}^2 \to \mathbb{R}^2$.

Proof

Writing (x,y), (u,v) for the standard coordinates on the domain and codomain respectively, consider the mappings:

(a) $u \equiv 0$, $v \equiv 0$

(b) $u \equiv 0$, $v = \mathrm{re}(b_k z^k)$

(c) $u = \mathrm{re}(a_l z^l)$, $v = \mathrm{re}(b_k z^k)$

where l,k are positive integers, $z=x+iy$, a_l, b_k are non-zero complex numbers, and, in case (c) $l > k$ or $l=k$ and $\mathrm{im}(a_l \bar{b}_k) \neq 0$.

These maps are all harmonic. Further the coordinates (x,y), (u,v) are admissible (1.4.8)(B), and therefore by giving l,k any positive integer values, we obtain all the cases of (2.2B) and thus realise all the types of singularities in theorem (2.2.11). ∎

2.4 Further Properties of the Singularities of a Harmonic Map

Let M,N be Riemann surfaces and let $f:M\to N$ be harmonic with respect to some (smooth) hermitian metric on N. We give some corollaries of theorem (2.2.11) concerning the behaviour of a harmonic map near a singularity. We first give some results noted by Lewy [Le1] and Heinz [He1].

Corollary 2.4.1

If $p\in M$ is a singular point for the mapping f, there exists a neighbourhood U of p such that $f|_U$ is not injective.

Proof

In theorem (2.2.11), we have case (1),(2),(3),(4). In case (1) our assertion is clear; in case (2), choosing U such that $\Sigma\cap U$ is a general fold, then the Jacobian J of f assumes positive and negative values on the opposite sides of $\Sigma\cap U$, thus $f|_U$ cannot be injective; in case (3), again J has opposite signs on opposite sides of the general folds, thus $f|_U$ cannot be injective; in case (4), if p is a branch point of order k-1 $(k>1)$, then (2.2.10) there exists a neighbourhood U of p such that each point of $f(U)$ has k preimages in U whence $f|_U$ is not injective. ∎

Corollary 2.4.2 [Le1], [He1]

If f is one-one in the open set $W\subset M$, then the Jacobian J of f is not zero at any point of W.

Proof

If the Jacobian of f were zero at $p\in W$, then by (2.4.1) we could choose a neighbourhood U of p such that $f|_U$ is not one-one contradicting the hypothesis. ∎

Corollary 2.4.3

The singular set $\Sigma\subset M$ is equal to the branch set $B\subset M$.

Proof

If $p\in\Sigma$, by (2.1.11) there exists a neighbourhood U of p such that $f|_U$ is not one-one, therefore $p\in B$. ∎

We now give some more propeties of the singularities of a harmonic map.

Corollary 2.4.4

Let p be a singular point and suppose for some neighbourhood U of p $\Sigma\cap U$ is a C^0 1-submanifold through p. Then J has opposite signs on opposite sides of $\Sigma\cap U$.

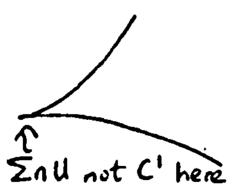
Proof

In theorem (2.2.11) we must have case (2) or case (3) with l-k=1. Thus $\Sigma\cap U$ is either a general fold or two general folds meeting at p. In either case (see note after (2.1.6)) J has opposite signs on opposite sides of $\Sigma\cap U$. ∎

Proposition 2.4.5

Let p be a singular point. Suppose, for some neighbourhood U of p, $\Sigma\cap U$ is a C^0 1-submanifold through p. Then $\Sigma\cap U$ is a C^1 1-submanifold through p.

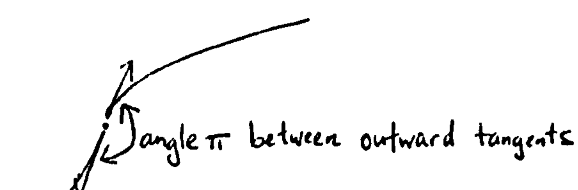
(I.e. $\Sigma\cap U$ can have no sharp corners.)



Example of prohibited form of $\Sigma\cap U$

Proof

If $\Sigma\cap U$ is a C^0 1-submanifold through p, then we can choose the neighbourhood U such that we have either case (2): $\Sigma\cap U$ is a general fold, or case (3): $\Sigma\cap U = \{p\}\cup$ two general folds. In the former case, $\Sigma\cap U$ is a C^∞ 1-submanifold and thus certainly a C^1 1-submanifold; in the latter case $\Sigma\cap U = \{p\}\cup$ two C^∞ 1-submanifolds with p a C^1 endpoint of each and with angle π between the outward tangents at p along each C^∞ 1-submanifold:



Case (3)

(see (2.2.9)) . Clearly the two C^{∞} 1-submanifolds combine to give a C^1 1-submanifold through, p. Thus $\Sigma \cap U$ is a C^1 1-submanifold through p. ∎

Proposition 2.4.6

An isolated singularity of f must be a branch point.

Proof

If $p \in M$ is an isolated singularity of f, we must have case (4) of theorem (2.2.11). Thus p is a branch point. ∎

2.5 Constrasts of the behaviour of Harmonic maps between Surfaces and the behaviour of other mappings

A Constrasts with non-harmonic maps between surfaces

Let M, N be Riemann surfaces and let $f: M \to N$ be harmonic with respect to some (smooth) hermitian metric on N. We here contrast some of the properties of the singularities of the harmonic map f with the behaviour of a non-harmonic map $k: \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \longmapsto (u, v)$ having a singularity at the origin. For exact statements of properties see theorem referred to.

Constrast 2.5.1 (see 2.2.2)

A zero p of df is isolated unless $df \equiv 0$ on a neighbourhood of p.

Constrast with

$k: u = x^2$, $v = 0$, or $u = x^2$, $v = x^2 y$; for both these mappings dk is zero on the line $x = 0$. ∎

Constrast 2.5.2 (see 2.2.4)

A bad singular point p (†) is isolated unless the Jacobian J is identically zero on a neighbourhood of p.

Constrast with

$u = x^3$, $v = y$. Here $J = 3x^2$, $dJ = (6x, 0)$, and so the y-axis consists of bad singularities. ∎

Constrast 2.5.3 (see 2.4.4)

J has opposite signs on opposite sides of any 1-submanifold portion of Σ

Constrast with

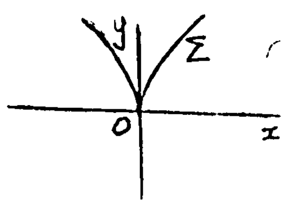
$u = x^3$, $v = y$; here $\Sigma =$ y-axis and $J = 3x^2$ has the same sign on each side of Σ. ∎

Constrast 2.5.4 (see 2.4.5)

A C^0 1-submanifold portion of Σ must be a C^1 1-submanifold.

Constrast with

$u = \frac{1}{3}x^3 - xy^3$, $v = y$; here $J = x^2 - y^3$ and thus Σ is the C^0 submanifold $x^2 = y^3$ which is not C^1-smooth at $(0,0)$:



(†) bad = not good!

Constrast 2.5.5 (see 2.4.3, 2.4.6)

The singular set $\Sigma \subset M$ equals the branch set $B \subset M$.

An isolated singular point must be a branch point.

Contrast with

$u = \frac{1}{3}x^3 + xy^2$, $v = y$; here $J = x^2 + y^2$. $(0,0)$ is an isolated singularity however it is easily seen that the map is a homeomorphism of the plane to the plane. Thus $\Sigma = (0,0)$, $B = \emptyset$, and $(0,0)$ is an isolated singularity which is not a branch point. ∎

B Constrasts with harmonic maps between manifolds of higher dimensions

We show here that our results for harmonic maps between surfaces do not generalise to higher dimensions. As usual, let M,N be Riemann surfaces and let $f:M \to N$ be harmonic with respect to some (smooth) hermitian metric on N. We constrast the behaviour of f with the behaviour of a harmonic map $k: \mathbb{R}^3 \to \mathbb{R}^3$, $(x,y,z) \mapsto (u,v,w)$.

Constrast 2.5.6 (c.f. 2.5.1)

A zero p of df is isolated unless $df \equiv 0$ on a neighbourhood of p.

Constrast with

$k: u = x^2 - y^2$, $v=0$, $w=0$; here dk is zero on the line $x=0$, $y=0$.

Constrast 2.5.7 (c.f. 2.5.2)

A bad singular point is isolated unless $J \equiv 0$ on a neighbourhood of p.

Constrast with

$u = xz + z^2 - y^2$, $v=yz$, $w=z$; here $J=z^2$ and the plane $z=0$ consists of bad singularities.

Constrast 2.5.8 (c.f. 2.5.3)

If $p \in \Sigma$, and for some neighbourhood U of p, $\Sigma \cap U$ is a C^0 1-submanifold i.e. a C^0 submanifold of codimension 1, J has opposite signs on opposite sides of $\Sigma \cap U$.

Constrast with

$u = xz+z^2-y^2$, $v = yz$, $w=z$; here $J=z^2$, Σ is the plane $z=0$, and J is positive on both sides of Σ .

Constrast 2.6.4 (c.f. 2.3.3)

The singular set $\Sigma\subset M$ equals the branch set $B\subset M$

Constrast with

$u = x^3-3xz^2+yz$, $v=y-3xz$, $w=z$; here $J=3x^2$ so the singular set is the plane $x=0$; however the mapping is a homeomorphism of $\underline{R}^3$ onto $\underline{R}^3$ (with inverse mapping $x = \sqrt[3]{u-vw}$, $y=v+3w\sqrt[3]{u-vw}$, $z=w$), thus its branch set is empty.

Remark

In particular the Lewy-Heinz theorem (2.4.2) does not hold in higher dimensions.

3. GLOBAL PROPERTIES OF HARMONIC MAPS

3.1 Global Description of A Harmonic Mapping According to Rank

Let M, N be connected Riemann surfaces, let $U \subset M$ be an open set. We shall say that the smooth mapping $f\colon M \to N$ between Riemann surfaces has maximum rank r on $U \subset M$ ($r = 0, 1, 2$) if $\max\{\text{rank } df(p)\colon p \in U\} = r$ (i.e. (1) rank $df(p) \leq r$ for all $p \in U$, (2) there exists $p \in U$ such that rank $df(p) = r$). Now let $f\colon M \to N$ be harmonic w.r.t. some hermitian metric on N. We describe f according to its maximum rank.

Theorem 3.1.1 [Sa] (Maximum Rank 0)

Let M, N be connected Riemann surfaces and let $f\colon M \to N$ be harmonic w.r.t. some hermitian metric on N. Suppose int $\Sigma_2 \neq \phi$, i.e. df is zero on an open set $U \subset M$. Then f maps M to a point. $\Sigma = \Sigma_2 = M$

Proof

We use Sampson's unique continuation theorem. Since df is zero on an open set U, f maps U to some point $q \in N$. Let f' be the constant mapping with $f'(U) = q$; then trivially f' is harmonic, f agrees with f' on U, and therefore by (1.4.5), f agrees with f' on their whole domain M. Thus f maps M to a point and $\Sigma = \Sigma_2 = M$. ∎

Corollary 3.1.2

If f has maximum rank 0 on an open set $U \subset M$, then f has maximum rank 0 on M. ∎

Remark 3.1.3

Theorem (3.1.1) is true for $f\colon M \to N$ a harmonic map between Riemannian manifolds of any dimensions. The same proof applies. ∎

Theorem 3.1.4 [Sa] (Maximum rank 1)

Let M, N be connected Riemann surfaces and let $f: M \to N$ be harmonic w.r.t. some hermitian metric on N.

(A) Suppose (a) f has maximum rank 1 on some non-empty open set $W \subset M$

or (b) $\operatorname{int} \Sigma \neq \phi$ and f is non-constant.

Then f maps M into a geodesic arc. $\Sigma = M$, Σ_2 = set of isolated points - the critical points of f as a mapping $M \to$ geodesic arc.

(B) Further, if M is compact, f maps M onto a closed geodesic γ.

(C) If M is compact of Euler characteristic $\chi(M)$, then $\operatorname{card}\Sigma_2 \leq -\chi(M)$, i.e. $f: M \to \gamma$ has at most $-\chi(M)$ critical points.

Proof

(A) If f has maximum rank 1 on some non-empty open set $W \subset M$, or equivalently, $\operatorname{int}\Sigma$ contains the non-empty open set W, then either $W \subset \Sigma_0$, in which case by (3.1.2) f is constant on M, and of maximum rank 0, or f has rank 1 at all points of the non-empty open set $V = W \setminus \Sigma_0$. Thus either hypothesis (a) or (b) implies f has rank 1 on some non-empty open set $V \subset M$. $J \equiv 0$ on V.

Now by proposition (2.2.6), if $p \in V$, there exists a neighbourhood U of p such that f(U) is a geodesic arc γ, and further in suitable normal coordinates (u,v) about f(p), f satisfies

$$u = 0 \qquad \Delta v = 0 \qquad \text{on } U.$$

Now prolong γ to the maximal geodesic through f(p). and along each simple portion introduce Fermi coordinates (u',v'). Then it is easily seen that in the Fermi coordinates, f still satisfies

$$u' = 0 \qquad \Delta v' = 0 \qquad \text{on } U$$

By the unique continuation theorem these equations have the same form on any larger open set which maps into the Fermi system.

By connectedness of M it follows that f maps the whole of M into a geodesic arc.

(B) Let f map M into the geodesic arc γ given by $\tilde{\gamma}:(\alpha,\beta) \to N$, where $-\infty \leqslant \alpha < \beta \gtreqless \infty$. Suppose γ is not a closed geodesic. Then $\tilde{\gamma}^{-1} \circ f : M \to R$ is by (2.2.6) a harmonic function. By the maximum principle for harmonic functions, $\tilde{\gamma}^{-1} \circ f$ must be constant, contradicting the hypotheses; therefore γ is a closed geodesic.

(C) Suppose M is compact of Euler characteristic $\chi(M)$. Firstly, $f:M \to N$ cannot be holomorphic or antiholomorphic, as its image is an arc. Thus we may apply (1.3.14) to conclude that df has at most $-\chi(M)$ zeros, i.e. $\mathrm{card}\Sigma_2 \leqslant -\chi(M)$. Note that a zero of df is a critical point of f considered as a mapping $M \to \gamma$. ■

Remark 3.1.5

Parts (A) and (B) of this theorem are true for $f:M \to N$ a harmonic map between Riemannian manifolds of any dimension (see [Sa]). ■

Corollary 3.1.6

If f has maximum rank 1 on some open set $U \subset M$, then f has maximum rank 1 on M. ■

Theorem 3.1.7 (Maximum rank 2)

Let M, N be connected Riemann surfaces and let $f:M \to N$ be harmonic w.r.t. some hermitian metric on N. Suppose f has maximum rank 2 on a non-empty open subset of M.

(A) Then the singular set Σ consists of:

(1) a locally finite number of disjoint general folds

(2) a discrete set of meeting points of general folds - such a point is an endpoint of class C^1 of an even number of disjoint general folds arranged at equal angles around p as explained in (2.2.9).

Conversely any endpoint of general folds must be such a meeting point.

(3) a collection of isolated points of Σ. These are branch points. Σ_2 is a discrete set consisting of all the branch points and some of the meeting points. Σ_1 consists of all the general folds and the rest of the meeting points.

(B) If further, M is compact, there are a finite number of general folds, meeting points and branch points.

(C) If M is compact and of Euler characteristic $\chi(M)$, df has at most $-\chi(M)$ zeros, i.e. card$\Sigma_2 \leq -\chi(M)$ unless f is holomorphic or antiholomorphic. If there is one or more general fold, f cannot be holomorphic or antiholomorphic.

Proof

(A) Let $p \in \Sigma$. Then there exists a neighbourhood U of p such that $\Sigma \cap U$ is described by case (1), (2), (3) or (4) of theorem (2.2.11). But if case (1) holds, i.e. $\Sigma \cap U = U$, then f has rank ≤ 1 on U, and by (3.1.2), (3.1.6), f has rank ≤ 1 on M, contradicting the hypotheses. Thus we have case (2), (3) or (4), i.e. $\Sigma \cap U$ is a general fold, or $\Sigma \cap U = \{p\} \cup$ even number of disjoint general folds, or $\Sigma \cap U = \{p\}$ and p is a branch point. The description of Σ follows noting that given $p \in M$, we can choose a neighbourhood U of p such that U contains a finite number of general folds, meeting points and branch points. Note also if p is an endpoint of a general fold it must be a meeting point. Also from (2.2.11), in case (2), $\Sigma_2 \cap U = \phi$, in case (3), $\Sigma_2 \cap U = \{p\}$ or ϕ, p being a meeting point of general folds, and in case (4), $\Sigma_2 \cap U = \{p\}$, p being a branch point.

Therefore Σ_2 is a discrete set consisting of all of the branch points and some of the meeting points. Since $\Sigma_1 = \Sigma \setminus \Sigma_2$ the description of Σ_1 follows.

(B) If M is compact, then, as remarked above, for each $p \in M$ we can choose a neighbourhood U of p such that U contains a finite number of general folds, meeting points and branch points. Such neighbourhoods form a cover of M. Extract a finite subcover. Then M is covered by a finite collection of open sets each containing a finite number of general folds, meeting points and branch points.

(C) If f is not holomorphic or antiholomorphic, $\operatorname{card}\Sigma_2 \leq -\chi(M)$ by (1.3.14). If there is at least one general fold, f cannot be holomorphic or antiholomorphic as it is well-known that the only singularities of such maps are branch points. ∎

Remark 3.1.8

If f has maximum rank 2 on some open set $U \subset M$ then f has maximum rank 2 on M (trivial). ∎

Combining (3.1.2), (3.1.6), (3.1.8) we have:

Theorem 3.1.9 (Constancy of maximum rank)

Let M, N be Riemann surfaces and let $f: M \to N$ be harmonic w.r.t. some hermitian metric on N. Then, if for some open set $U \subset M$, f has maximum rank r on U ($r = 0, 1, 2$), then f has maximum rank r on M.

Proof

From (3.1.2), 3.1.6), (3.1.8). ∎

Remark

In fact we can assert that if on some open set $U \subset M$, f has maximum rank r ($r = 0, 1, 2$), then f has maximum rank r on any open subset of M. ∎

Remark

(1) This theorem appears not to generalise to harmonic maps between higher dimensional C^{∞} Riemannian manifolds, though the author knows no counter-example.

(2) If M, N are C^ω manifolds and $f:M \to N$ is a not necessarily harmonic C^ω map, then constancy of maximum rank follows from the principle of analytic continuation. ∎

3.2 Harmonic maps as ramified coverings

Let M, N be connected Riemann surfaces, and let $f:M \to N$ be a smooth map. Recall (2.1.19) that we say f is ramified at $p \in M$ if p has a neighbourhood U such that $f:U \setminus p \to N$ is a local homeomorphism.

Definition 3.2.1

We say $f:M \to N$ is a ramified covering or f defines M as a ramified covering surface of N if $f:M \to N$ is ramified at every point $p \in M$. ∎

Notes

(1) Such a map defines M as a ramified covering surface of N in the sense of Ahlfors and Sario [A-S §1.20] (see proof of 2.1.20).

(2) As a special case, $f:M \to N$ is a ramified covering if it is a local homeomorphism at every point $p \in M$. Following Ahlfors and Sario we call such a map a smooth (†) covering. A smooth covering has empty branch set. ∎

Proposition 3.2.2 (Characterisation of ramified coverings)

(A) A smooth map $f:M \to N$ is a ramified covering

$\Leftrightarrow$ its branch set $B \subset M$ is empty or consists of isolated points

$\Leftrightarrow$ its branch set is empty or consists of branch points.

(B) The singular set of a ramified covering $f:M \to N$ contains all the branch points of f but may contain singular points which are not branch points.

Proof

(A) As remarked after (2.1.19), f is ramified at p iff either p is an isolated point of the branch set B of f or $p \notin B$. Therefore f is a ramified covering if and only if its branch set is empty or consists of isolated points. By (2.1.25) p is an isolated point of the branch set if and only if it is a branch point.

(B) A branch point is certainly a singularity since $B \subset \Sigma$; consider

(†) Note "smooth" is not used here in the sense of "infinitely differentiable".

however the mapping $f:\mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x^3,y)$. This is a ramified covering (in fact it is a homeomorphism), has empty branch set, but has non-empty singular set $\Sigma = \{(x,y): x = 0\}$. Thus the singular set of this ramified covering contains points which are not branch points.$\blacksquare$

We now discuss the question: "When is a harmonic map a ramified covering?"

Proposition 3.2.3 (Characterisation of harmonic ramified coverings)

(A) Let $f:M \to N$ be a non-constant holomorphic or antiholomorphic map. Then f is a ramified covering. The singular set consists of all the branch points of f.

(B) Let $f:M \to N$ be a non-constant map which is harmonic with respect to some hermitian metric on N. Suppose we know:

either (1) Σ_1 is empty

or (2) f has no general folds and is of maximum rank 2;

then f must be a ramified covering. Further the singular set of f consists solely of branch points.

(C) If $f:M \to N$ is a ramified covering, we can choose a new complex structure for M such that f is holomorphic.

Proof

(A) It is well-known that the only singularities of a holomorphic or antiholomorphic map $f:M \to N$ are branch points. Since $B \subset \Sigma$, f is a ramified covering by (3.2.2)(A).

(B) Let $f:M \to N$ be a non-constant harmonic mapping. Note first that if f has maximum rank 1, by (3.1.4), $\Sigma = M$, Σ_2 = set of isolated points, therefore $\Sigma_1 = \Sigma \setminus \Sigma_2$ is non-empty.

Now suppose (1): Σ_1 is empty. Then by the preceding remark, f must have maximum rank 2. By (3.1.7) $\Sigma = \Sigma_2$ = discrete set of branch points.

Suppose, instead, (2): f has no general folds and is of maximum rank 2. Then by (3.1.7) $\Sigma = \Sigma_2$ = discrete set of branch points.

Thus in either case, Σ consists solely of branch points. Since $B \subset \Sigma$, the branch set consists solely of branch points, and hence by (3.2.2)(A) f is a ramified covering.

(C) This is a result of Ahlfors and Sario [A-S64B] Note that we do not claim that the new complex structure gives the same smooth structure on M as the original complex structure.$\blacksquare$

B The degree of a mapping and the scope of ramified coverings

Let $f: M \to N$ be a smooth map between oriented smooth connected manifolds of the same dimension, with M compact. Then at each ordinary point $p \in M$, $df_p: TM_p \to TN_{f(p)}$ is a linear isomorphism of oriented vector spaces.(*) Define the sign of df_p to be +1 or −1 according as df_p preserves or reverses orientation. For any regular value $q \in N$, define

$$\deg(f;q) = \sum_{p \in f^{-1}(q)} \operatorname{sign} df_p$$

It can be shown [Mi p.27 ff.] that the integer deg(f;q) does not depend on the choice of regular value q; thus we may simply write deg(f). This integer is called the Brouwer degree of the mapping f. It can further be shown that deg(f) depends only on the smooth homotopy class of f. Note that if $f: M \to N$ is not surjective, deg(f) = 0 (proof: choose $q \in N \setminus f(M)$).

Now suppose M, N are Riemann surfaces, with M compact. Then M, N are certainly oriented smooth connected manifolds of equal dimension 2, and thus we may define the degree of a mapping $f: M \to N$ as above. We now investigate the degree of a ramified covering.

Proposition 3.2.4

Let M, N be Riemann surfaces with M compact, and let $f: M \to N$ be a ramified covering. Then each regular value $q \in N$ has precisely $|\deg(f)|$ inverse images, where deg(f) is the Brouwer degree of f. In particular, deg(f) cannot be zero, f must be surjective, and N must be compact.

(*) $df_p = df(p)$ = derivative of f at p.

Proof

Since a ramified covering has a discrete singular set, it is clear that either sign $df_p \geq 0$ for all $p \in M$ or sign $df_p \leq 0$ for all $p \in M$. If $q \in N$ is a regular value:

$$\deg(f) = \deg(f;q) = \sum_{p \in f^{-1}(q)} \text{sign } df_p$$
$$= \pm(\text{number of inverse images of } q)$$

(+ if sign $df_p \geq 0$ for all $p \in M$, - if sign $df_p \leq 0$ for all $p \in M$) (†) Therefore q has precisely $|\deg(f)|$ inverse images.

Suppose now $\deg(f) = 0$. Then every regular value has empty inverse image. This is clearly not true for a ramified covering. Therefore $\deg(f)$ cannot be zero. It follows that $|\deg(f)| \geq 1$, so that each regular value has at least 1 inverse image. But by definition each singular value has at least 1 inverse image. Therefore f is surjective. $N = f(M)$ is the continuous image of a compact space and is therefore compact. ∎

We can include singular values as well, as follows:

Proposition 3.2.5 (Degree of a Ramified Covering)

Let M, N be Riemann surfaces, with M compact, and let $f:M \to N$ be a ramified covering. Then each point $q \in N$ has precisely $|\deg(f)|$ inverse images provided we count each inverse image according to its multiplicity (2.1.21). (Recall a branch point of order $k - 1$ has multiplicity k, an ordinary point has multiplicity 1.)

Proof

Ahlfors and Sario call a ramified covering $f:M \to N$ complete if for every $q \in N$ there exists a neighbourhood V of q such that $f^{-1}(\bar{V})$ is compact. Clearly if M is compact, any ramified covering is complete. The proposition now follows from [A-S 21B]. ∎

Note $|\deg(f)|$ is often called the number of sheets of f.

(†) Note if f is holomorphic, $df_p \geq 0$ for all p, thus $\deg(f) \geq 0$.

We ask whether there are any ramified coverings of degree ± 1:

Proposition 3.2.6

Let M, N be Riemann surfaces with M compact, and let $f:M \to N$ be a ramified covering. Then if $|\deg(f)| = 1$, f must be a homeomorphism.

Proof

For each point $q \in N$ has precisely 1 inverse image. ■

We shall use this result in §3.3.

C Hurwitz' Formula

If M, N are compact Riemann surfaces and $f:M \to N$ is a ramified covering, the ramification index r of f is defined as the sum of the orders of all the branch points of f.

Theorem 3.2.7 (Hurwitz' Formula)

Let M, N be compact Riemann surfaces and let $f:M \to N$ be a ramified covering of degree d. Let r denote the ramification index of f. Then

$$\chi(M) + r = |d|\chi(N) \tag{3.2A}$$

Proof

See for example Stoïlow [St]. ■

This formula tells us how many branch points a ramified covering has. We shall use it in §5.2. Note that if $r = 0$ - i.e. we have a smooth covering - Hurwitz' formula reduces to the well-known formula: $\chi(M) = |d|\chi(N)$.

D Harmonic ramified coverings of the 2-sphere

Let M be a compact Riemann surface and let S be the Riemann sphere. Many harmonic maps of M into S can be given by constructing holomorphic maps $M \to S$. By [E-S p118] such maps are harmonic w.r.t. any choice of hermitian metric on M or S. Such maps may be found by constructing meromorphic maps $M \to \underline{C} \cup \infty$ using Riemann-Roch theory. This is dealt with in detail in §5.1. For the moment we content ourselves with listing the results:

Theorem 5.1.13

Let M be a compact surface of genus g. Then there exist holomorphic maps $M \to S$ of degree n for:

all $n \geq 1$, if $g = 0$;

all $n \geq 2$, if $g = 1, 2$;

all $n \geq g + 1$ and at least one n, $2 \leq n \leq g$, if $g \geq 3$.

Proof

See §5.1. ∎

If $g = 0$, so that M is the Riemann sphere, holomorphic maps of degree $n \geq 1$ are given explicitly by $z \mapsto c\, z^n$ $(c > 0)$ where the Riemann sphere is identified with the extended complex plane $\{z\colon z \in \mathbb{C} \cup \infty\}$ in the usual way.

If $g = 1$, then M is a torus and the theorem tells us harmonic maps of degree ≥ 2 exist. It is an open question whether there exists a harmonic map of degree 1 of the torus to the 2-sphere.

By (3.3.1) such a map would have general folds, and by (1.3.16) its derivative would have no zeros.

We note that Smith [Sm p.95] constructs explicit examples of harmonic maps of the torus into the standard 2-sphere exhibiting fold lines and collapse lines. These maps all have degree 0.

3.3 Harmonic Maps with General Folds

A Occurrence

Let M, N be connected Riemann surfaces and let $f:M \to N$ be harmonic w.r.t. some hermitian metric on N. Suppose f has maximum rank 2. Then Theorem (3.1.7) describes the singular set of f. If $\Sigma_1 = \phi$ we saw (3.2.3) that f is a ramified covering; we now deal with the case $\Sigma_1 \neq \phi$. In this case f has general fold lines (3.1.7). We first describe the occurrence of such maps.

Theorem 3.3.1 (Need for general folds)

Let M, N be compact Riemann surfaces of unequal genera. Let $f:M \to N$ be harmonic w.r.t. some hermitian metric on N, and suppose $\deg(f) = \pm 1$. Then f has general folds.

Proof

Firstly f must be surjective, otherwise we would have $\deg(f) = 0$. Therefore f must have maximum rank 2, otherwise, by (3.1.1), (3.1.4), f is certainly not surjective. By (3.2.3), if f has no general folds, then it is a ramified covering. By (3.2.7), $\deg(f) = \pm 1$ implies f is a homeomorphism, which contradicts the hypothesis that genus(M) $\neq$ genus(N). ∎

Corollary 3.3.2 (Occurrence of general folds)

Given any two compact Riemann surfaces with genus(M) > genus(N) > 0 there exist harmonic maps $M \to N$ with general folds.

Proof

Give N a hermitian metric with non-positive curvature. By the existence theorem (1.1.11) of Eells and Sampson, any homotopy class of maps $M \to N$ contains a harmonic one. Since genus(M) > genus(N) there exist homotopy classes of degree ± 1. Therefore there exist harmonic maps of degree ± 1. By (3.3.1), any such harmonic map has general folds. ∎

B The structure of the general folds

In §2.1 we saw that a general fold can be a fold line, a collapse line or can contain cusps or good singular points of order r, $3 \leq r \leq \infty$, in which case it is neither a fold line nor collapse line. However note the

Lemma 3.3.3

If $\gamma \subset M$ is a general fold for the smooth map $f: M \to N$, we can find a fold line or a collapse line $\subset \gamma$.

Proof

Let $p \in \gamma$. Then p is a good singular point of some order r, $1 \leq r \leq \infty$. If $r = 1$, then by (2.1.11) p is interior to a fold line; if $r > 1$, then either there exists a neighbourhood U of p such that $U \cap \gamma$ consists of good singular points of order > 1, or given any neighbourhood U of p, $U \cap \gamma$ contains at least one good singular point p' of order 1. In the former case, by definition (2.1.7) and notes after (2.1.9), the directional derivative $\nabla_w f(q) = 0$ for $q \in U \cap \gamma$, w tangent to γ at q, and therefore f is constant on $U \cap \gamma$ which is thus a collapse line; in the latter case, by (2.1.11) some neighbourhood of p' is a fold line. Thus we can find either a fold line or a collapse line contained in γ. ∎

Theorem 3.3.4 (Occurrence of folds and collapse lines)

Let M, N be Riemann surfaces and let $f: M \to N$ be harmonic w.r.t. some hermitian metric on N. Suppose $\Sigma_1 \neq \phi$. Then there exists at least one fold line or one collapse line.

Proof

For if $\Sigma_1 \neq \phi$, then by (3.1.7) f has at least one general fold which by (3.3.3) contains at least one fold line or collapse line. ∎

Remarks

It would be nice to assert that having removed a discrete set of points from Σ consisting of meeting points and good singular points of order > 1 which are not collapse points, then Σ consists of fold lines, collapse lines and branch points. We could then think of f roughly as a "branched and folded" covering of Tucker [Tu]. However there is no guarantee that the set of good singular points of order > 1 which are not collapse points is a discrete set. ∎

C Tucker's Formula [Tu]

Suppose M, N are compact and $f:M \to N$ can be triangulated as a "branched and folded covering" in the sense of Tucker (see [Tu]); then Tucker gives a formula relating the amount of branching and folding to the degree of the map, viz:

$$\chi(M) + \sum_e e\chi^{[e]} = |d|\chi(N)$$

Here e denotes the "exceptionability" of a point of M - 0 for an ordinary point, k - 1 for a branch point of order k, - 1 for a fold line; $\chi^{[e]}$ denotes the Euler characteristic of the "subcomplex" of points of exceptionability e, and d denotes the degree of f.

Note if f has no general folds but only branch points, Tucker's formula reduces to Hurwitz' formula. We shall not use Tucker's formula in any essential way.

D The number of general folds of a harmonic map into the flat torus.

We are interested in putting an upper bound on the number of general folds, meeting points, etc., of a harmonic map $f:M \to N$ between Riemann surfaces. Using special techniques, in §5.2 we can derive such bounds in the case N = flat torus. We quote the result:

Theorem 5.2.14

Let $f:M \to T$ be a harmonic map from a compact Riemann surface of genus g into the flat torus $T = S^1 \times S^1$ (equipped with its standard metric).

Then one of the following holds:

(1) f is constant;

(2) f maps M onto a closed geodesic and has at most $2g - 2$ critical points (cf. 3.1.4(C));

(3) f is a ramified covering with ramification index $2g - 2$;

(4) f has singular set Σ consisting of at most $2g - 2$ branch points, $(2g - 2)(6g - 6)$ general folds and $6g - 6$ meeting points.

<u>Proof</u>

See §5.2. ∎

3.4 Questions of Monotonicity

Here we answer questions relating to when a harmonic map is monotone or quasi-monotone in the sense of P. T. Church [Ch 2]. We restrict attention to smooth maps f between compact Riemann surfaces M, N.

A Monotone Maps

Definition 3.4.1 [Ch 2]

$f:M \to N$ is called monotone if for each $q \in N$, $f^{-1}(q)$ is connected (or empty).∎

Church proves the following result concerning the occurrence of monotone maps:

Proposition 3.4.2 [Ch 3; 4.1]

Let $f:M \to N$ have points $p \in M$ where f is of rank 2. Then f is monotone onto if and only if: $\deg f = 1$ and sign $df_p \geq 0$ $\forall p \in M$, or $\deg f = -1$ and sign $df_p \leq 0$ $\forall p \in M$. (See §3.3B for definition of deg f, etc.)

Using this we prove:

Theorem 3.4.3

Let M, N be compact Riemann surfaces and let $f:M \to N$ be harmonic with respect to some hermitian metric on N. Then f is monotone onto if and only if f is a diffeomorphism.

Proof

If f is a diffeomorphism, then it is monotone by definition, and onto.

Conversely, suppose f is monotone onto. Then f must have maximum rank 2, otherwise, by (3.1.1), (3.1.4), f would not be onto. Now, without loss of generality, assume that $\deg f = 1$ and sign $df_p \geq 0$ $\forall p \in M$. We assert that f can have no general folds. For J and therefore sign df has opposite signs on opposite sides of any general fold (2.1.5) contradicting sign $df_p \geq 0$ $\forall p \in M$. By (3.2.3)(B) it follows that f is a ramified covering of degree 1 whose singular set consists of a finite number of branch points. But since, by (3.2.5), each point q of N has precisely one inverse image counting according to multiplicity, M can have no branch points. Thus $\Sigma = \phi$ and f is a diffeomorphism. ∎

Remark 3.4.4

If f is not harmonic, a maximum rank 2 map which is monotone and onto need not even be a homeomorphism; for example we can construct a smooth map of the sphere to the sphere which collapses a line segment to a point but is otherwise a diffeomorphism. ∎

Remark 3.4.5

Church [Ch 2; 1.3] shows that for any smooth map $f:M \to N$, if $q \in N \smallsetminus f(\Sigma_2)$ then $f^{-1}(q)$ has at least $|\deg f|$ components, therefore any monotone map has degree ± 1 or 0. ∎

B Quasi-Monotone Maps

We see that monotonicity is a severe restriction on a map and thus we are led to consider a less severe restriction:

Definition 3.4.6 [Ch]

$f:M \to N$ is called quasi-monotone if for each connected open set $V \subset N$ and component U of $f^{-1}(V)$, $f(U) = V$. ∎

Monotone maps are clearly quasi-monotone; Church [Ch 4; p.380] further shows that if dim B = 0 (where, as usual, B denotes the branch set of f (see §2.1)) then f is quasi-monotone - thus ramified coverings (§3.2) are quasi-monotone. We now show the converse is true for a harmonic map:

Theorem 3.4.7

Let M, N be compact Riemann surfaces and let f be harmonic with respect to some hermitian metric on N, and of maximum rank 2. Then f is quasi-monotone if and only if f is a ramified covering.

Proof

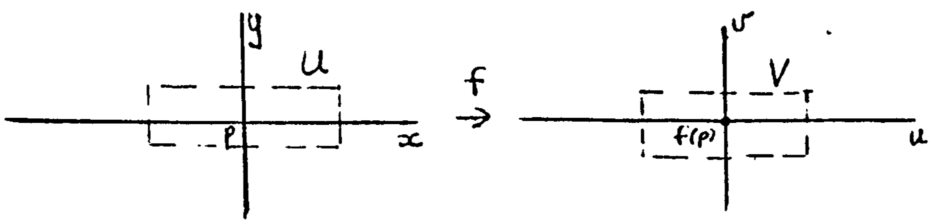
As remarked above, if f is a ramified covering, then f is quasi-monotone.

Conversely, suppose f is quasi-monotone and of maximum rank 2. Suppose $\Sigma_1 \neq \phi$. By (3.3.4), there is (a) a fold line or (b) a collapse line.

(a) Suppose $p \in$ fold line. Then by (2.1.11) we can choose smooth coordinates (x,y), (u,v) about p,f(p) such that f has the form:

$$u = x^2, \quad v = y.$$

Now choose V to be the inside of a small rectangle centred on f(p) as shown:



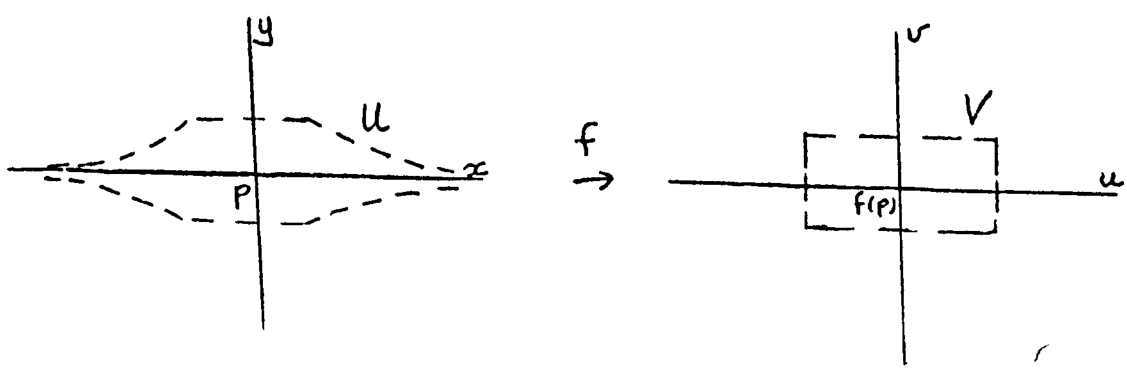
Then V is a connected open set. Let U = component of $f^{-1}(V)$ containing p. Then U is the inside of a small rectangle centred on p as shown above. This maps into the right-hand half of V. Thus by definition f is not quasi-monotone.

(b) Suppose $p \in$ collapse line. Then by (2.1.18) we can choose smooth coordinates (x,y), (u,v) centred on p,f(p) such that f has the form:

$$u = xy, \quad v = y.$$

Now, if V is a small enough neighbourhood of f(p), let U = the component of $f^{-1}(V)$ containing p, then (see diagram below)

$$f(U) \subset V \setminus \{(u,v): u \neq 0, \ v = 0\}$$



Thus $f(U) \neq V$ and so f is not quasi-monotone.

We have thus shown that $\Sigma_1 = \phi$. Hence, by (3.2.3)(B), f must be a ramified covering. ∎

Remark 3.4.8

Again, this theorem is by no means true for non-harmonic maps f. In fact Church [Ch 4; (3)] shows that any map with $\dim B \leq 1$ and $\operatorname{sign} df_p \geq 0 \; \forall p \in M$ or $\operatorname{sign} df_p \leq 0 \; \forall p \in M$ is quasi-monotone. ∎

3.5 Convex Functions and the Image of a Harmonic Map

For this section let M, N be connected smooth Riemannian manifolds of dimensions m, n and let $f:M \to N$ be harmonic. We recall (1.2.7) that a subset $D \subset N$ is called convex supporting if every compact subset of D has an open neighbourhood in N which supports a strictly convex function. We recall also Gordon's result (1.2.7) that, if M is compact, f cannot have image in a convex supporting subset of N unless constant. We shall here list some corollaries of this result which we shall compare with results obtained by an entirely different method in §4.3.

Corollary 3.5.1 [Go]

Let $f:M \to N$ be harmonic, M compact. Then each point $b_o \in N$ has a convex supporting neighbourhood V_{b_o}. $f(M)$ cannot lie in any such neighbourhood unless f is constant.

Proof

Every point $b_o \in N$ has a neighbourhood V_{b_o} which supports the convex function $k(b) = \sum_{i=1}^{n} (u^i)^2$ where $(u^1, \ldots, u^n)$ are any b_o-centred smooth coordinates for N. ∎

Corollary 3.5.2

Let $f:M \to N$ be harmonic, M compact, N complete. Suppose N has non-positive sectional curvatures. Then f(M) cannot lie within a simply connected open subset D of N unless f is constant.

Proof

Let $b_o \in D$ be a fixed point. Since N is complete there is a geodesic from any point $b \in D$ to b_o. Since D is simply connected, this geodesic is unique. By an argument of Bishop and O'Neill [BO'N4.1(2)] we can show the squared geodesic distance of b from b_o is strictly convex. Thus D is convex supporting. ∎

Corollary 3.5.3

Let $f:M \to S^2$ be harmonic, M compact (S^2 = Euclidean 2-sphere). Then

$f(M)$ cannot lie within an open hemisphere of S^2 unless f is constant.

Proof

An open hemisphere of S^2 is convex supporting [Go p. 436]. ∎

Corollary 3.5.4

Let $f: M \to N$ be harmonic, M compact, N complete. Suppose N has strictly negative sectional curvatures. Then for each closed geodesic $\gamma \subset N$ there exists an open set U with $\gamma \subset U \subset N$ such that $f(M)$ cannot lie in U unless f maps into γ.

Proof

Take U to be a tubular neighbourhood diffeomorphic by the exponential map to $\gamma \times D^{n-1}$ where D^{n-1} is the $(n - 1)$-dimensional open disc; then [B-O'N 4.1(1)] distance squared from γ is strictly convex on U, and thus U is convex supporting. ∎

4. THE GAUSS-BONNET FORMULA AND THE IMAGE OF A HARMONIC MAP

4.1 Description of the Image of A C^{ω} Map

A Analytic and Semi-Analytic Sets

Let M be a connected C^{ω} manifold of dimension m. We shall define semi-analytic sets following Lojasiewicz [Lo 1, 2].

Definition 4.1.1

A subset $A \subset M$ is called a C^r submanifold of dimension k ($r = 1, 2, \ldots, \infty, \omega$) if for every point $p \in A$ there is a neighbourhood U of p and C^r functions $r_1, \ldots, r_{m-k}$ on U such that:

(i) $U \cap A = \bigcap_{j=1}^{m-k} \{q \in U : r_j(q) = 0\}$,

(ii) the derivatives $dr_1(p), \ldots, dr_{m-k}(p) \in L(TM_p, \mathbb{R})$ are linearly independent. ∎

It is easily seen using the inverse function theorem that this definition agrees with the definition of 1-submanifold in §2.1A. We shall generally be interested in C^{ω} submanifolds, which are often called analytic submanifolds.

Definition 4.1.2 [Lo 1]

A subset A of M is called semi-analytic if for every point $p \in A$ there is a neighbourhood U of p and real analytic functions $r_1, \ldots, r_s$ on U such that $A \cap U$ is a finite union of finite intersections of sets of the form:

(a) $\{q \in U : r_j(q) > 0\}$, (b) $\{q \in U : r_j(q) = 0\}$

A is called analytic if we only use sets of the form (b). ∎

It is easy to see [Lo 1, 2] that the union of any locally finite family and the intersection of any finite family of semi-analytic sets is semi-analytic, as is the complement of a semi-analytic set.

Definition 4.1.3 [Lo 2]

We say that a point p of a semi-analytic set A is regular of dimension k if, for some neighbourhood U of p, $A \cap U$ is an analytic submanifold of dimension k. We define the dimension of a semi-analytic set to be the maximum dimension of its regular points.

Thus an analytic submanifold of dimension k is a semi-analytic set (in fact an analytic set) of dimension k; note also that the subset of points of a semi-analytic set which are regular of dimension k forms an analytic submanifold.

Lemma 4.1.4

Let γ be the image of the finite open interval $(0,1) \subset \mathbb{R}$ under a real-analytic mapping $F:(0,1) \to M$. Suppose that $\frac{dF}{dt} \neq 0 \ \forall\, t \in (0,1)$ and that F extends to a real-analytic mapping $F:[0,1] \to M$ on the closed interval $[0,1]$. Then γ is a semi-analytic subset of M of dimension 1.

Proof

(First Step) Let $t_o \in (0,1)$. We show that there exists a neighbourhood V of t_o in (0,1) such that F(V) is semi-analytic of dimension 1.

For, by hypothesis, $\frac{dF}{dt}(t_o) \neq 0$. Therefore, by the implicit function theorem, there exists a neighbourhood V of t_o in (0,1) such that $F|_V$ is an analytic isomorphism onto an analytic 1-submanifold. Then certainly F(V) is semi-analytic of dimension 1. Note V can be considered as a neighbourhood of t_o in [0,1].

(Second Step) Let $t_o = 0$ or 1. We show that there exists a neighbourhood V of t_o in [0,1] such that F(V) is semi-analytic of dimension 1.

For without loss of generality we may assume $t_o = 0$. Then c.f. [W p.392] w.r.t. any C^ω coordinates (x,y) for M centred on $F(t_o)$, the components of dF can be written near $t_o = 0$ in the form:

$$\frac{dx}{dt} = t^r\,\Phi(t) \qquad \frac{dy}{dt} = t^s\,\Psi(t) \qquad (\dagger)$$

where $r,s \in \{0,1,2,\ldots\}$ and Φ, Ψ are C^ω functions defined near $0 \in \mathbb{R}$ with $\Phi(0) \neq 0$, $\Psi(0) \neq 0$. By continuity we may choose $\varepsilon > 0$ such that $\Phi(t) \neq 0$, $\Psi(t) \neq 0\ \forall\, t \in [0,\varepsilon)$, whence

$$\frac{dx}{dt} \neq 0,\ \frac{dy}{dt} \neq 0 \quad \forall t \in (0,\varepsilon)$$

We now claim $F|_{[0,\varepsilon]}$ is injective.

For suppose $x(t_1) = x(t_2)$ for some $0 \leq t_1 < t_2 \leq \varepsilon$. Then by Rolle's Theorem, $\exists\, t$, $t_1 < t < t_2$ such that $\frac{dx}{dt}(t) = 0$. We thus have a contradiction, and thus $F|_{[0,\varepsilon]}$ is injective.

We now show $F(0,\varepsilon)$ is semi-analytic of dimension 1.

For let $t_o \in (0,\varepsilon)$, $p_o = F(t_o)$. We show that there exists a neighbourhood U of p_o in M such that $F(0,\varepsilon) \cap U$ is semi-analytic. Firstly, just as in (First Step), there exists a neighbourhood V' of t_o in $(0,\varepsilon)$ such that $F(V')$ is an analytic submanifold of dimension 1. Secondly, consider $F([0,\varepsilon] \setminus V')$. It is a compact set, and since $F|_{[0,\varepsilon]}$ is injective, it does not contain p_o. Therefore there exists a neighbourhood U of p_o in M such that $F([0,\varepsilon] \setminus V') \cap U = \phi$ and therefore $F((0,\varepsilon) \setminus V') \cap U = \phi$. It easily follows that $F(0,\varepsilon) \cap U = F(V') \cap U$. Since $F(V')$ is an analytic submanifold of dimension 1, it follows that $F(0,\varepsilon) \cap U$ is a semi-analytic set of dimension 1. Now this is true for a neighbourhood U of any point $p_o \in F(0,\varepsilon)$. Therefore $F(0,\varepsilon)$ is semi-analytic of dimension 1.

It follows that $F[0,\varepsilon) = F(0) \cup F(0,\varepsilon)$ is semi-analytic of dimension 1, as we have the union of two semi-analytic sets.

(Third Step) We have thus shown that, for any $t_o \in [0,1]$, there exists a neighbourhood V of t_o in $[0,1]$ such that $F(V)$ is semi-analytic of dimension 1. Such neighbourhoods form an open covering of $[0,1]$. Extract a finite subcover $V_1,\ldots V_k$. Then $F[0,1] = F(V_1) \cup \ldots \cup F(V_k)$

($\dagger$) We have omitted the possibility that $\frac{dx}{dt}$ or $\frac{dy}{dt}$ might be identically zero (Note they cannot both be identically zero by the hypotheses). The argument needs trivial modification in this case.

is the finite union of semi-analytic sets of dimension 1, and therefore is a semi-analytic set of dimension 1. ∎

B Semi-analyticity of the singular set

Let M, N be connected two-dimensional C^ω manifolds and let $f:M \to N$ be a C^ω map. We have already met the singular set

$$\Sigma = \{p \in M: \dim \ker df(p) > 0\}$$

and its subsets

$$\Sigma_1 = \{p \in M: \dim \ker df(p) = 1\}$$

$$\Sigma_2 = \{p \in M: \dim \ker df(p) = 2\}$$

Lemma 4.1.5

(1) Σ, (2) Σ_2 are analytic sets; (3) Σ_1 is a semi-analytic set.

Proof

(1) Let $p \in \Sigma$, take U to be a coordinate neighbourhood, then $\Sigma \cap U = \{q \in U: J(q) = 0\}$. Since J is a real-analytic function on U, this shows Σ is analytic.

(2) Let $p \in \Sigma_2$, again take U to be a small enough coordinate neighbourhood; then $\Sigma_2 \cap U = \bigcup_{i=1,2} \{q \in U: f_i^\alpha(q) = 0\}$ where f_i^α are the components of the derivative w.r.t. any C^ω local coordinates about p, f(p). Since each f_i^α is real-analytic, Σ_2 is an analytic set.

(3) $\Sigma_1 = \Sigma \setminus \Sigma_2 = (M \setminus \Sigma_2) \cap \Sigma$. This shows Σ_1 is semi-analytic. ∎

We now introduce the set $\Sigma_{1,1} \subset M$.

$\Sigma_{1,1} = \{p \in \Sigma_1$: p is regular of dimension 1 and $\nabla_w f(p) = 0$ for w tangent to Σ_1 at p$\}$

Here $\nabla_w f(p)$ denotes the directional derivative of f at p in direction w. Note that the assumption that Σ_1 is regular at p ensures Σ_1 has a tangent at p.

Lemma 4.1.6

$\Sigma_{1,1}$ is a semi-analytic set.

Proof

Let $p \in M$. Choose C^ω coordinates (x,y), (u,v) for M, N, centred on p, f(p) with the x-axis *along* Σ. Then

$$\nabla_w f(q) = 0 \Leftrightarrow \frac{\partial u}{\partial x}(q) = 0 \text{ and } \frac{\partial v}{\partial x}(q) = 0$$

Thus on a coordinate neighbourhood U of p:

$$\Sigma_{1,1} \cap U = \{q \in \Sigma_1 : q \text{ is regular of dimension } 1\} \cap U \cap \{q: \frac{\partial u}{\partial x}(q) = 0\} \cap \{q: \frac{\partial v}{\partial x}(q) = 0\}$$

This clearly exhibits $\Sigma_{1,1}$ as a semi-analytic set.∎

C An Analytic Triangulation of M

Let M be a countable (†) C^ω surface and let $\{A_\alpha\}$ be a collection of subsets of M. A (locally finite, finite) analytic triangulation of M is a (locally finite, finite) simplicial complex K together with a homeomorphism $T: |K| \to M$ (onto M) such that:

(4.1B) for any $\sigma \in K$, $T(\sigma)$ is an analytic submanifold of M and $T|_\sigma : \sigma \to T(\sigma)$ is an analytic isomorphism. (*) (Note: a C^r triangulation ($r = 0, 1, \ldots, \infty$) of M can be defined similarly.)

The triangulation is said to be compatible with $\{A_\alpha\}$ if

(4.1C) for any $\sigma \in K$ and any A_α, either $T(\sigma) \subset A_\alpha$ or $T(\sigma) \subset M \setminus A_\alpha$.

Lojasiewicz proves [Lo 1] that given a locally finite collection of semi-analytic subsets $\{A_\alpha\}$ of a countable analytic surface M, there exists a locally finite triangulation compatible with $\{A_\alpha\}$. If M is compact this triangulation is finite. Using this we prove:

Theorem 4.1.7 (Triangulation of M)

Let M, N be real-analytic surfaces, M countable, and let $f: M \to N$ be a real-analytic map. Then there exists a locally finite analytic triangulation $T: |K| \to M$ compatible with the disjoint subsets $M \setminus \Sigma$, $\Sigma_1 \setminus \Sigma_{1,1}$, $\Sigma_{1,1}$, Σ_2. Thus the image $T(\sigma)$ of each simplex $\sigma \in K$ is contained in precisely one of the sets $M \setminus \Sigma$, $\Sigma_1 \setminus \Sigma_{1,1}$, $\Sigma_{1,1}$, Σ_2.

(†) M countable means M has countable basis of open sets - often termed 2^{nd} countable. (*) σ will always denote an open simplex.

Proof

By (4.1.5), (4.1.6), we see $M \setminus \Sigma$, $\Sigma_1 \setminus \Sigma_{1,1}$, $\Sigma_{1,1}$, Σ_2 are all semi-analytic; therefore we may triangulate by Lojasiewicz. Note also these four sets are disjoint and have union M. ∎

For simplicity of notation we now identify a simplex $\sigma \in K$ with its image $T(\sigma) \subset M$.

Proposition 4.1.8

The triangulation of M in (4.1.7) has the property that if σ is a simplex of M, $f|_\sigma$ has constant rank.

Proof

(1) If σ is a r-simplex of $M \setminus \Sigma$, $f|_\sigma$ has constant rank r.

(2) If σ is a 2-simplex of $\Sigma_1 \setminus \Sigma_{1,1}$, $f|_\sigma$ has constant rank 1.

(3) If σ is a 1-simplex of $\Sigma_1 \setminus \Sigma_{1,1}$, $f|_\sigma$ has constant rank 1.

(4) If σ is a 1-simplex of $\Sigma_{1,1}$, $f|_\sigma$ has constant rank 0.

(5) If σ is any simplex of Σ_2, $f|_\sigma$ has constant rank 0.

(6) If σ is any 0-simplex, $f|_\sigma$ has constant rank 0.

It is easily seen from the definitions that we have covered all the possibilities. ∎

Remarks 4.1.9

(1) Our triangulation (4.1.7) could be used to give a description of Σ as analytic 1-submanifolds meeting at points like that in (3.1.7) for C^∞ harmonic maps.

(2) Theorem (4.1.7) does not generalise to smooth maps between smooth surfaces, e.g. let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x,y) \mapsto (e^{-1/x^2} \sin\frac{1}{x}, y)$, then it is easily seen that no locally finite (C^∞) triangulation of M compatible with Σ exists (for Σ consists of lines parallel to the y-axis with an infinite number of lines in any neighbourhood of the y-axis). ∎

D An Analytic Triangulation of $f(M)$

Again let M, N be connected real-analytic surfaces, M countable, and let $f:M \to N$ be a real-analytic map.

Lemma 4.1.10 (Constancy of maximum rank)

f has maximum rank r on $M \Leftrightarrow f$ has maximum rank r on any open subset U of M ($r = 0, 1, 2$).

Proof

If f has maximum rank r on U ($r = 0, 1, 2$), then all the $(r + 1)\times(r + 1)$ minors of the matrix for df in any local coordinates vanish on U. By analytic continuation these minors vanish on the whole of M, therefore f has maximum rank r on M. ∎

Remarks

(1) This is of course true for M, N real-analytic manifolds of any dimension.

(2) For surfaces, we saw that this is true also for C^∞ harmonic maps (3.1.9). It is not true, in general, for a C^∞ map. ∎

Now let $f:M \to N$ have maximum rank 2 and be proper. (†)

Lemma 4.1.11

$f(M)$ is a semi-analytic set.

Proof

Triangulate M in any way so that, for each simplex σ of M, $f|_\sigma$ has constant rank. One way is given in (4.1.7). We shall now show that the image $f(\sigma)$ of each simplex $\sigma \in M$ is a semi-analytic set.

Let σ be a 2-simplex of M. Then by (4.1.10), $f|_\sigma$ must have rank 2. It easily follows that $f(\sigma)$ is an open subset of N, so that $f(\sigma)$ is an analytic submanifold of dimension 2, and is therefore semi-analytic.

Let σ be a 1-simplex of M. Then if $f|_\sigma$ has rank 0 $f(\sigma)$ is a point which is certainly a semi-analytic set. Suppose instead $f|_\sigma$ has rank 1.

(†) $f:M \to N$ is proper iff $f^{-1}(W)$ is compact for every compact set $W \subset N$.

Then the triangulation map T gives an analytic isomorphism $T: (0,1) \to \sigma$. Consider the composition $\tilde{f}:(0,1) \xrightarrow{T} \sigma \xrightarrow{f} N$. Then $\tilde{f}$ is a C^ω map. It extends to a C^ω map $\tilde{f}:[0,1] \xrightarrow{T} \bar{\sigma} \xrightarrow{f} N$, and since $f|_\sigma$ has rank 1, $\frac{d\tilde{f}}{dt} \neq 0 \; \forall \; t \in (0,1)$. We may thus apply (4.1.4) to show $f(\sigma)$ is a semi-analytic set.

Lastly let σ be a 0-simplex of M. Then $f(\sigma)$ is a point, which is certainly a semi-analytic set.

Thus we have proved that if σ is any simplex of M, $f(\sigma)$ is a semi-analytic set.

Now $f(M) = \bigcup_{\sigma \in M} f(\sigma)$, thus $f(M)$ is the union of semi-analytic sets. We now show this is a locally finite union: for let $b \in f(M)$ and let V be a neighbourhood of b with $\bar{V}$ compact. Then since f is <u>proper</u> $f^{-1}(\bar{V})$ is compact and therefore meets a finite number of simplexes of M. Hence $\bar{V}$(and hence V) meets only a finite number of $\{f(\sigma)\}$. Thus the collection $\{f(\sigma)\}$ is locally finite. Thus $f(M)$ is the locally finite union of semi-analytic sets, and is therefore semi-analytic.∎

<u>Lemma 4.1.12</u>

$\partial f(M)$ - the topological boundary of $f(M)$ - is a semi-analytic set.

<u>Proof</u>

The boundary of a semi-analytic set is semi-analytic [Lo 2].∎

<u>Proposition 4.1.13</u>

Let M, N be countable real analytic surfaces and let $f:M \to N$ be real-analytic, of maximum rank 2, and proper.

(A) There exists a locally finite analytic triangulation $\mathcal{U}$ of N compatible with $f(M)$, $\partial f(M)$. If N is compact (so that $M = f^{-1}(N)$ must be also*) $\mathcal{U}$ is a finite triangulation.

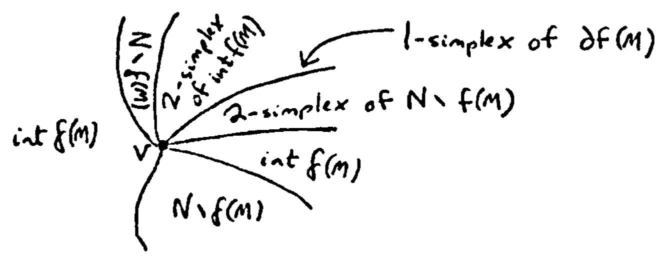
* Since f is proper

Suppose further that M, N are compact, then:

(B) in any finite analytic triangulation $\mathcal{U}$ of N compatible with f(M), ∂f(M), f(M) is a pure subcomplex of dimension 2, and ∂f(M) is empty or is a pure subcomplex of dimension 1.

(Here a <u>pure</u> subcomplex of dimension r is one such that every s-simplex s < r is a face of an r-simplex).

(C) Any vertex (= 0-simplex) v of ∂f(M) is the face of an even number of 1-simplexes of ∂f(M). Any such 1-simplex of ∂f(M) is the face of two 2-simplexes of N one of which lies in int f(M) and the other of which lies in N∖f(M). Thus the 1-simplexes of ∂f(M) which "meet" at a vertex v of ∂f(M) divide a neighbourhood of v like a pie into an even number of sectors alternately in int f(M), N∖f(M):



<u>Example of triangulation near a vertex</u>

<u>Proof</u>

(A) Since N is countable and f(M), ∂f(M) are semi-analytic, such a triangulation exists by Lojasiewicz. If N is compact the triangulation must be finite.

(B) (First Step) We show f(M) is a subcomplex of dimension 2. Firstly f(M) is a closed collection of simplices and is therefore a subcomplex of N. Further, since f has maximum rank 2, it is easy to see that f(M) contains at least one 2-simplex. Therefore, f(M) is a subcomplex of dimension 2.

(Second Step) We now show f(M) is a <u>pure</u> subcomplex. Note first that every simplex of N lies either entirely in f(M) or entirely in N∖f(M). Let s ∈ f(M) be a 0-simplex. We show s is a face of some 1-simplex of f(M). Suppose not. Then all the 1-simplexes of N in St(s) (= the <u>star</u> of s) are simplexes of

$N \setminus f(M)$. We claim this implies that all the 2-simplexes of N in $St(s)$ are also in $N \setminus f(M)$. For suppose not. Then at least one 2-simplex of N in $St(s)$ would be in $f(M)$ and all its 1-faces would also be in $f(M)$, contradicting the fact that all the 1-simplexes of N in $St(s)$ are simplexes of $N \setminus f(M)$. Thus all the 1- and 2-simplexes of N in $St(s)$ lie in $N \setminus f(M)$. Therefore, $f(M) \cap St(s) = s$. Thus if $s = f(p)$, then the neighbourhood $f^{-1}\{St(s)\}$ of p maps to the single point s. By (4.1.10) this means f has maximum rank 0 contradicting the hypotheses. We have thus shown if s is a 0-simplex of $f(M)$, then s is the face of some 1-simplex of $f(M)$. A similar argument shows that any 1-simplex of $f(M)$ is the face of some 2-simplex of $f(M)$. We conclude that $f(M)$ is a pure subcomplex of dimension 2.

i (Third Step) We show $\partial f(M)$ is either empty or is a pure subcomplex of dimension 1. For by definition of topological boundary, $\partial f(M)$ can contain no open subsets of N. Therefore $\partial f(M)$ can contain no 2-simplexes. Further, $\partial f(M)$ is closed, therefore $\partial f(M)$ is either empty or is a subcomplex of dimension $\leqslant 1$.

Now let s be a 0-simplex of $\partial f(M)$. We shall show that s is the face of a 1-simplex of $\partial f(M)$. For suppose not. Then all the 1-simplexes in $St(s)$ lie in $N \setminus \partial f(M)$. Further all the 2-simplexes of $St(s)$ lie in $N \setminus \partial f(M)$, since $\partial f(M)$ contains no 2-simplexes. Therefore all the simplexes of $St(s)$ lie in $N \setminus \partial f(M)$ (except s). Now $N \setminus \partial f(M) = \text{int } f(M) \cup \{N \sim f(M)\}$. Since the triangulation of N is compatible with $f(M)$, $\partial f(M)$, every simplex of $N \setminus \partial f(M)$ lies in either int $f(M)$ or $N \setminus f(M)$. Consider the 2-simplexes of $St(s)$.

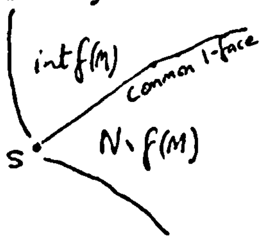
Suppose they are all contained in int $f(M)$. Then all their 1-faces, in particular all the 1-simplexes of $St(s)$, are contained in $f(M)$, and therefore in int $f(M)$(*) Therefore $St(s) \setminus s \subset \text{int } f(M)$. Since $s \subset \partial f(M) \subset f(M)$ this implies $s \subset \text{int } f(M)$, contradicting the

(*) since we can choose a neighbourhood of any point in such a 1-simplex contained wholly in $f(M)$:

hypotheses.

Suppose, on the other hand, that all the 2-simplexes of $St(s)$ are contained in $N \setminus f(M)$. Then $s \in f(M)$ but $St(s) \cap f(M)$ contains no 2-simplexes. As before this contradicts the maximum rank 2 of f.

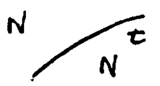
Therefore there is at least one 2-simplex of $St(s)$ contained in int $f(M)$, and at least one 2-simplex of $St(s)$ contained in $N \setminus f(M)$. It is easy to see that we can choose a pair of such 2-simplexes



which have a common 1-face. This face clearly lies in $\partial f(M)$ and has s as 0-face. Thus we have shown that the arbitrary 0-simplex s of $\partial f(M)$ is the face of some 1-simplex of $\partial f(M)$. Therefore $\partial f(M)$ is a pure subcomplex of dimension 1.

(C) Note first that since our triangulation of N is compatible with $f(M)$, $\partial f(M)$, every simplex lies in just one of the sets: int $f(M)$ $(= f(M) \setminus \partial f(M))$, $\partial f(M)$, $N \setminus f(M)$.

Now let v be a 0-simplex of $\partial f(M)$. By (3) above v is the face of at least one 1-simplex of $\partial f(M)$. Let t be any such 1-simplex. Then it is certainly the face of precisely two 2-simplexes of N:



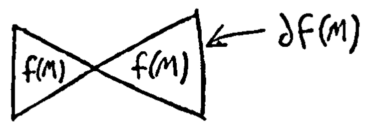
Then, since $\partial f(M)$ contains no 2-simplexes, each 2-simplex lies in either int $f(M)$ or $N \setminus f(M)$. But $t \subset \partial f(M) \subset f(M)$ and $f(M)$ is a pure subcomplex of dimension 2. Therefore, at least one of the 2-simplexes lies in $f(M)$, and, therefore, in int $f(M)$. Suppose they both lie in int $f(M)$. Then clearly $t \subset$ int $f(M)$, contradicting hypothesis. Therefore one of the 2-simplexes lies in int $f(M)$ and the other lies in $N \setminus f(M)$.

Thus if v is a 0-simplex of $\partial f(M)$, we have shown that any 1-simplex of $\partial f(M)$ with 0-face v is the face of two 2-simplexes, one of which lies in int $f(M)$ and one of which lies in $N \setminus f(M)$. Thus the totality of 1-simplexes of $\partial f(M)$ with 0-face v divide a neighbourhood of v like a pie into a number of sectors alternately in int $f(M)$, $N \setminus f(M)$ - there must obviously be an even number of such sectors and thus an even number of 1-simplexes of $\partial f(M)$ with 0-face v. ☐

4.2 The Gauss-Bonnet Inequality for a Harmonic Map

A The Gauss-Bonnet Formula

Let M, N be compact real-analytic surfaces N oriented and let N have a real-analytic Riemannian metric. Let f be a real-analytic map of maximum rank 2. Triangulate N with a finite analytic triangulation $\mathcal{U}$ compatible with f(M), ∂f(M) - this is possible by (4.1.13). We wish to apply the Gauss-Bonnet formula to the region f(M). However, it is not necessarily true that f(M) is bounded by disjoint closed arcs - for example ∂f(M) might be in the form of a figure 8:



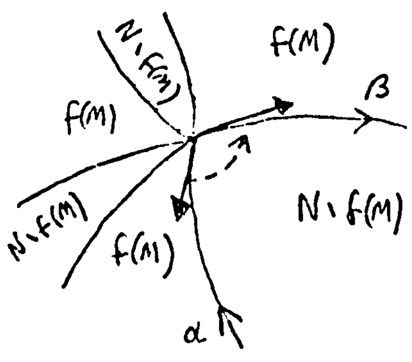
In order to apply the Gauss-Bonnet formula we need to describe how we traverse the boundary ∂f(M). Firstly, orient f(M) as a subset of N and orient each 1-simplex of ∂f(M) as a boundary arc of f(M). We now partition all the 1-simplexes into subsets which form closed curves.

Let α,β be 1-simplexes of ∂f(M) oriented as above; then we say α precedes β or β succeeds α, written $\alpha \to \beta$ if:

(1) α, β have a common vertex (0-simplex) p;

(2) α is oriented such that p is its final point, β is oriented such that p is its initial point (i.e. α "goes into" p and β "comes out" from p);

(3) If we rotate in a positive sense a tangent vector t at p from a position along the outward tangent along α at p, the first 1-simplex that t becomes tangent to is β, i.e. β is the next arc in the positive sense of rotation from α at p.

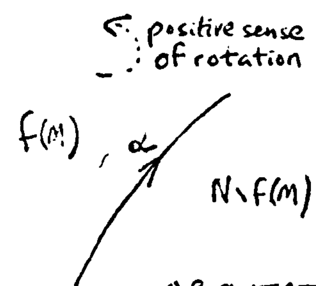


Example

Here α precedes β

positive sense of rotation

→ shows outward tangents along α, β



ORIENTATION CONVENTIONS FOR DIAGRAMS

It is easily seen that each 1-simplex of $\partial f(M)$ has a unique predecessor and unique successor (use 4.1.13(C)). Note also if $\alpha \to \beta$, the "sector" between α and β lies in $N \setminus f(M)$.

Now write $\alpha \sim \gamma$ if there is a sequence $\alpha, \beta, \ldots, \gamma$ such that $\alpha \to \beta \to \ldots \to \gamma$ or $\gamma \to \ldots \to \beta \to \alpha$. We see that ~ is an equivalence relation which partitions the 1-simplexes of $\partial f(M)$ into subsets, each subset forming a closed piecewise-analytic curve. Thus the boundary of f(M) consists of a finite number of closed piecewise-analytic curves $c_1, \ldots, c_r$. Of course these need not be <u>disjoint</u> curves. Nevertheless we shall show that we may apply the Gauss-Bonnet formula to the region f(M).

<u>Proposition 4.2.1</u>

Let M, N be compact analytic surfaces, N oriented, let N have an analytic Riemannian metric and let $f: M \to N$ be a real-analytic map. Then we may apply the Gauss-Bonnet formula to $f(M) \subset N$ in the form:

$$\text{(4.2A)} \qquad \int_{f(M)} K{*}1 = 2\pi\chi(f(M)) - \sum_{\substack{\text{all} \\ \text{1-simplexes} \\ \alpha \subset \partial f(M)}} \int_\alpha k\, ds - \sum_{\substack{\text{all} \\ \text{1-simplexes} \\ \alpha \subset \partial f(M)}} \theta(\alpha)$$

Here, K is the curvature of N at a point of f(M)

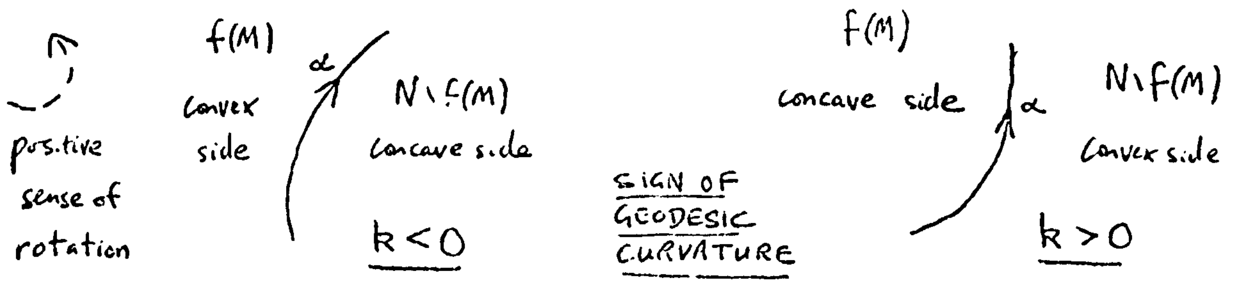
*1 is the volume 2-form of N

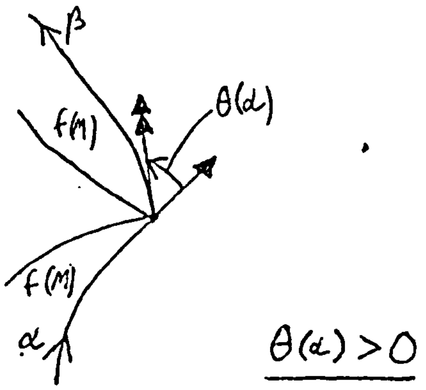
$\chi(f(M))$ is the Euler characteristic of f(M)

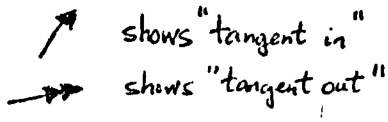
k is the signed geodesic curvature of the analytic arc α

$\theta(\hat{\alpha})$ is the signed "corner" angle $-\pi \leq \theta(\hat{\alpha}) \leq \pi$ from the tangent of α going into b ("tangent in") to the tangent of β coming out of b ("tangent out"), where α, β meet at b and α precedes β.

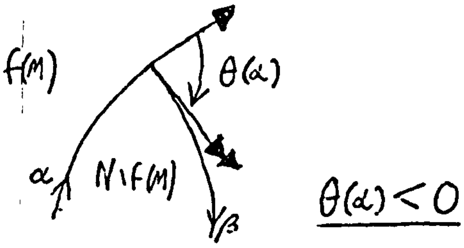
(Our sign conventions are standard - see for example Hicks [Hi]).





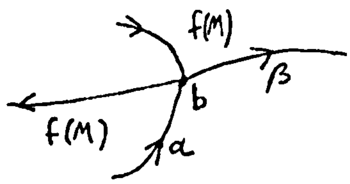


SIGN OF CORNER ANGLE

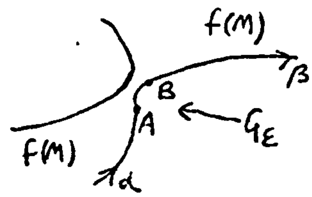


Proof

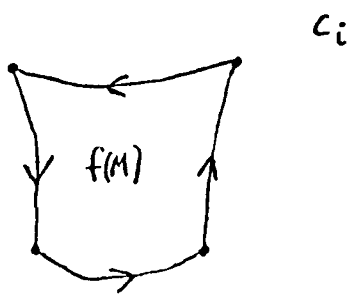
Partition the 1-simplexes of $\partial f(M)$ into subsets forming closed piecewise analytic curves $c_1, \ldots, c_r$. These curves may intersect at vertices (0-simplexes) of $\partial f(M)$. We shall apply a construction of Kreysig [Kr] which will deform these curves $c_1, \ldots, c_r$ into <u>disjoint</u> curves and allow us to apply Gauss-Bonnet. Let α, β be two 1-simplexes of the curve c_i meeting at b with $\alpha \to \beta$. By a <u>geodesic circle</u> centre



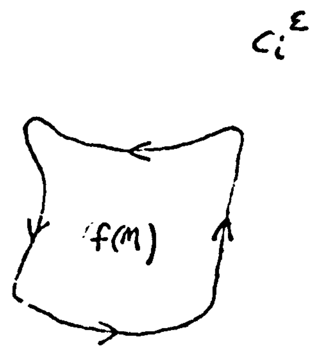
REPLACING A VERTEX BY GEODESIC CIRCLES



$c \in N$ radius r we shall mean a closed curve given in normal coordinates (u,v) centred on c by $u^2 + v^2 = r^2$, parametrised by $\phi \to (r \cos \phi, r \sin \phi)$. Replace the vertex by an arc G_ε of a geodesic circle of small radius $\varepsilon > 0$ whose tangents at the endpoints A, B of G_ε coincide with those of α, β respectively. Carry out this construction for every pair of 1-simplexes α, β of $f(M)$ such that $\alpha \to \beta$. Thus we replace $f(M)$ by a slightly larger region R_ε whose boundary consists of a finite number of <u>disjoint</u> C^1 curves $c_1^\varepsilon, \ldots, c_r^\varepsilon$, each curve c_i^ε approximating the original curve.



REPLACING THE BOUNDARY CURVES



Applying the Gauss-Bonnet formula [Kr p.209] to the region R_ε:

$$(4.2B) \qquad \int_{R_\varepsilon} K{*}1 = 2\pi\chi(R_\varepsilon) - \sum_{i=1}^{r} \int_{c_i} k \, ds$$

We now wish to let $\varepsilon \to 0$. First note $\chi(R_\varepsilon) = \chi(f(M))$, for $f(M)$ is a strong deformation retract of R_ε. Secondly, consider the contribution from an arc G_ε of a geodesic circle. If the arc is given in normal coordinates by $u = \varepsilon \cos\phi$, $v = \varepsilon \sin\phi$ ($\phi_0 \leq \phi \leq \phi_1$), then $k = \frac{1}{\varepsilon}$, $ds = \varepsilon \, d\phi$, and $\phi_1 - \phi_0 \to \theta(\alpha)$ as $\varepsilon \to 0$. Hence:

$$\lim_{\varepsilon \to 0} \int_{G_\varepsilon} k \, ds = \lim_{\varepsilon \to 0} \int_{\phi_0}^{\phi_1} \frac{1}{\varepsilon} \varepsilon \, d\phi = \lim_{\varepsilon \to 0} (\phi_1 - \phi_0) = \theta(\alpha)$$

Hence, for the curve c_i^ε:

$$\lim_{\varepsilon \to 0} \int_{c_i^\varepsilon} k \, ds = \sum_{\alpha \subset c_i} \int_\alpha k \, ds + \sum_{\alpha \subset c_i} \theta(\alpha)$$

Hence letting $\varepsilon \to 0$ in (4.2B):

$$\int_{f(M)} K{*}1 = 2\pi\chi(f(M)) - \sum_{\alpha \subset \partial f(M)} \int_\alpha k \, ds - \sum_{\alpha \subset \partial f(M)} \theta(\alpha)$$

as claimed. $\blacksquare$

B Applying the Gauss-Bonnet formula to a Harmonic Map

Let M, N now be compact Riemann surfaces, N with chosen real analytic hermitian metric. Let $f{:}M \to N$ be a harmonic map of maximum rank 2. Since f is real-analytic (1.1.4), we can apply the Gauss-Bonnet formula to $f(M)$ as in proposition (4.2.1). We now get further information by making use of Sampson's maximum principle for the harmonic map f.

Lemma 4.2.2

In (4.2A), (1) at all points of all the 1-simplexes α of $\partial f(M)$, $k \leq 0$; (2) for all 1-simplexes $\alpha \subset \partial f(M)$, $\theta(\alpha) \leq 0$.

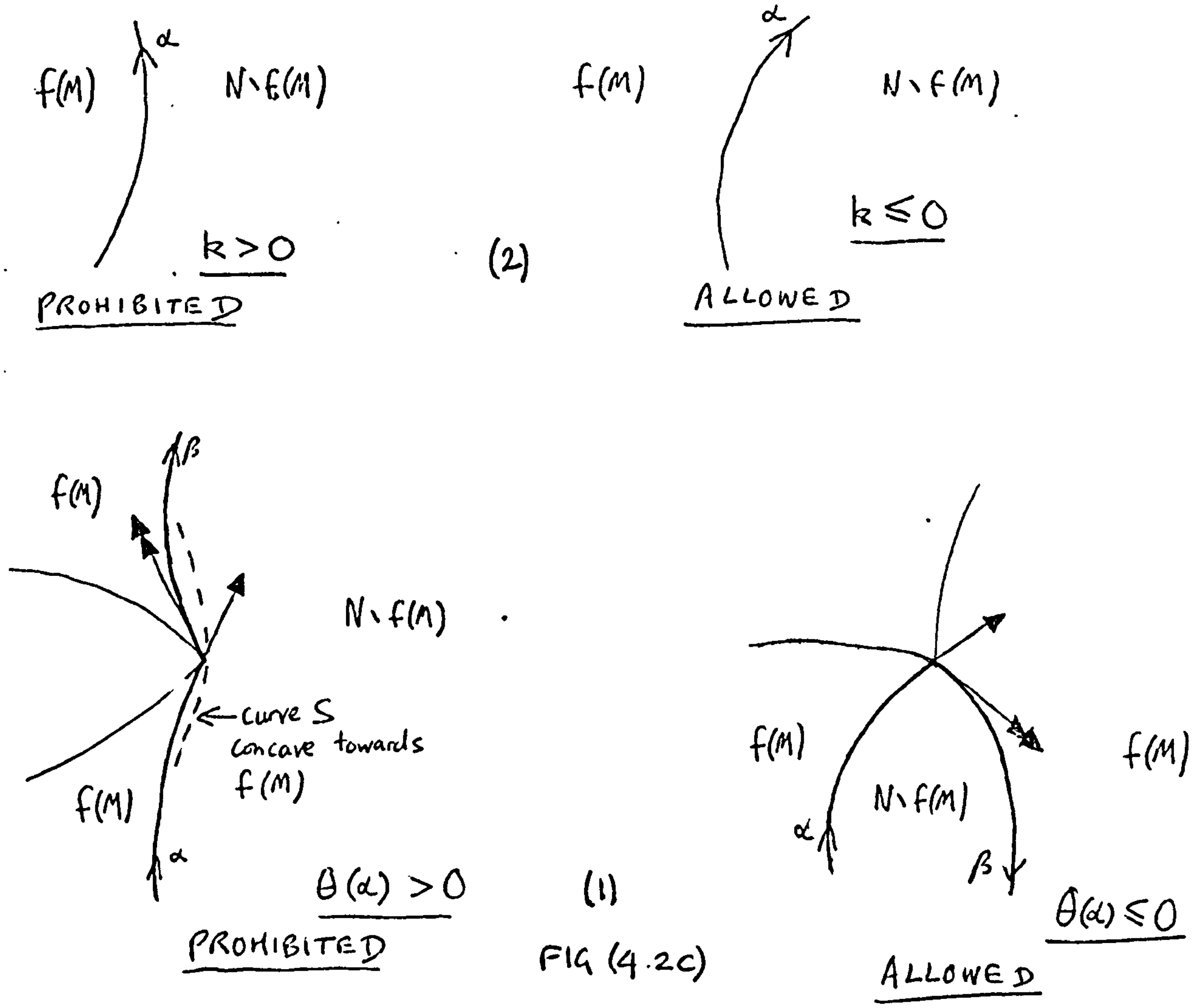


FIG (4.2c)

<u>Proof</u>

(1) Let b be a point on a 1-simplex $\alpha \subset \partial f(M)$. By Sampson's maximum principle (1.2.12), f(M) must lie to the convex side of α. Therefore $k \leq 0$.

(2) Let $b = f(p)$ be a vertex of $\partial f(M)$, and let α, β be 1-simplexes meeting at b with $\alpha \to \beta$. As remarked earlier, the sector between α and β lies in $N \setminus f(M)$. Now if $\theta(\alpha) > 0$, then (see the figure (4.2C)) we can draw a curve S such that f(M) lies to the concave side of S. Thus some neighbourhood of p maps to the concave side of S. This is prohibited by the maximum principle (1.2.12) and therefore $\theta(\alpha) \leq 0$. ∎

<u>Main Theorem 4.2.3 (Gauss-Bonnet inequality)</u>

Let M, N be compact Riemann surfaces, N with chosen real-analytic hermitian metric. Let $f: M \to N$ be a non-constant harmonic map. Then the total curvature $K(f(M))$ of $f(M)$ is related to the Euler characteristic $\chi(f(M))$ of $f(M)$ by the inequality:

(4.2D) $\quad K(f(M)) \geq 2\pi\chi(f(M))$

Proof

If f is of maximum rank 2, we apply Proposition (4.2.1) and Lemma (4.2.2) and the inequality follows.

If f is of maximum rank 1, then, by (3.1.4)(B), $f(M)$ is a closed geodesic. Thus $K(f(M)) = 0$, $\chi(f(M)) \leq 0$ and we have (4.2D). ∎

Remarks

(1) If f is a constant mapping $K(f(M)) = 0$, $\chi(f(M)) = 1$, and the inequality (4.2D) does not hold.

(2) If f is surjective, then $f(M) = N$, $\partial f(M)$ is empty; thus in (4.2D) the terms $\sum_\alpha \int_\alpha k\, ds$ and $\sum_\alpha \theta(\alpha)$ are zero, hence (4.2D) becomes:

(4.2E) $\quad K(N) = 2\pi\chi(N)$

which is a well-known formula for a compact Riemann surface.

(3) If N is not compact, there are many Riemannian surfaces which satisfy the Cohn-Vossen inequality:

$$K(N) < 2\pi\chi(N)$$

This shows that our inequality (4.2D) does not always hold, for it does not hold in the case $M = N$, f = identity map. Thus our inequality does not apply to the non-compact case. ∎

4.3 Consequences of the Inequality

Let M, N be compact Riemann surfaces, M with chosen real-analytic hermitian metric. We list some consequences of Proposition (4.2.3) and compare with results from convex functions in §3.5.

Corollary 4.3.1

Let $f:M \to N$ be harmonic, M, N compact. Then $f(M)$ cannot have Euler characteristic ≥ 1 (in particular, cannot be contractible) unless its total curvature $\geq 2\pi$, or f is constant.

Proof

For if f is non-constant and $\chi(f(M)) \geq 1$, then by (4.2.3), $K(f(M)) \geq 2\pi$. ∎

Remarks

Cf. (3.5.1) - these results are distinct: we can construct convex supporting subsets D which have curvature $> 2\pi$ (e.g. if $N = S^2$, $D = S^2 \setminus$ half a great circle - see [Go]). It is not known to the author whether contractible neighbourhoods V exist which are not convex supporting but have total curvature $< 2\pi$.

Corollary 4.3.2

Let $f:M \to N$ be harmonic, M, N compact. Suppose N has non-positive curvature. Then $f(M)$ cannot have Euler characteristic ≥ 1 (in particular, $f(M)$ cannot be contractible) unless f is constant.

Proof

If f is non-constant, then since $K \leq 0$, by (4.2.3), $\chi(f(M)) \leq 0$. ∎

Remarks

Cf. (3.5.2).

Corollary 4.3.3

Let $f:M \to S^2$ be harmonic, M compact (S^2 = Euclidean 2-sphere). Then, if $\chi(f(M)) \geq 1$ (e.g. $f(M)$ is contractible), $f(M)$ must cover half or more of the surface area of S^2 unless f is constant.

Proof

If f is non-constant and $\chi(f(M)) > 0$, then by (4.2.3), $K(f(M)) \geq 2\pi$. Since S^2 has constant curvature and total curvature 2π, this implies $f(M)$ covers half or more of the surface area of S^2. ∎

Remarks

Cf. (3.5.3). ∎

Corollary 4.3.4

Let $f:M \to N$ be harmonic, M, N compact. Suppose N has strictly negative curvature on a dense subset of N. Then, if $\chi(f(M)) \geq 0$, f must be either constant or map M onto a closed geodesic. (In particular, $f(M)$ cannot be a tubular neighbourhood about a closed curve on N.)

Proof

For if f has maximum rank 2, $K(f(M)) < 0$, therefore, from (4.2.3), $\chi(f(M)) < 0$. ∎

Remarks

Cf. (3.5.4). ∎

4.4 Redundant Folds

A The Theorems

If we think of a harmonic map $f:M \to N$ between surfaces as describing how a rubber sheet M is stretched over a surface N, we would not expect the rubber sheet to lie on N with a pair of "opposite" folds which could be "pulled out". We shall now develop two results which show that our intuition is correct, provided N has negative curvature. We shall use the Gauss-Bonnet theorem.

Let M, N be compact Riemann surfaces of genus g, h > 0 respectively, let N have a chosen smooth hermitian metric. Then M, N are C^1 diffeomorphs of spheres with g, h handles respectively. We define a handle as follows:

Definition 4.4.1

A <u>handle</u> H of M (or N) is a smooth submanifold-with-boundary of M (or N) which is C^1-diffeomorphic to $S'\times I = S'\times[0,1]$, i.e. there exists a C^1-diffeomorphism, C^1 up to the boundary:

$$(S'\times I,\ S'\times\partial I) \to (H,\ \partial H) \quad \blacksquare$$

Let $f:M \to N$ be a harmonic map such that a handle $H\subset M$ maps into a handle $K\subset N$, thus f restricts to a smooth map, smooth up to the boundary: $f:H \to K$. Our first result states that the handle H cannot map into the handle K with a pair of opposite folds.

Theorem 4.4.2

Let $f:M \to N$ be a smooth map which restricts to a map of handles:

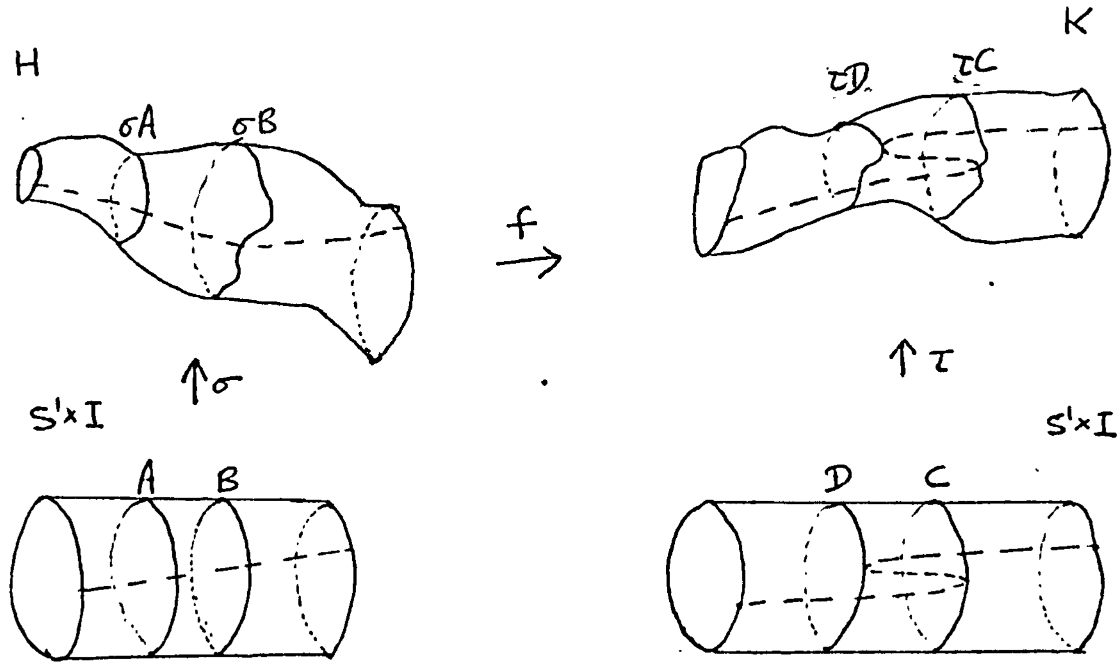
$$f:\ (H,\ \partial H) \to (K,\ \partial K)$$

Suppose this map is such that there exist C^1-diffeomorphisms, C^1 up to the boundary:

$$\sigma:\ (S'\times I,\ S'\times\partial I) \to (H,\ \partial H),\quad \tau:(S'\times I,\ S'\times\partial I) \to (K,\ \partial K),$$

such that the singular set of $f|_H$ consists of two closed fold lines

σA, σB where $A = S^1\times\{a\}$, $B = S^1\times\{b\}$, $0 < a < b < 1$, whose images are two closed C^1 curves τC, τD respectively where $C = S^1\times\{c\}$, $D = S^1\times\{d\}$, $0 \leq d < c \leq 1$. Then if the curvature of N is strictly negative on a dense subset of N, f cannot be harmonic.



Proof

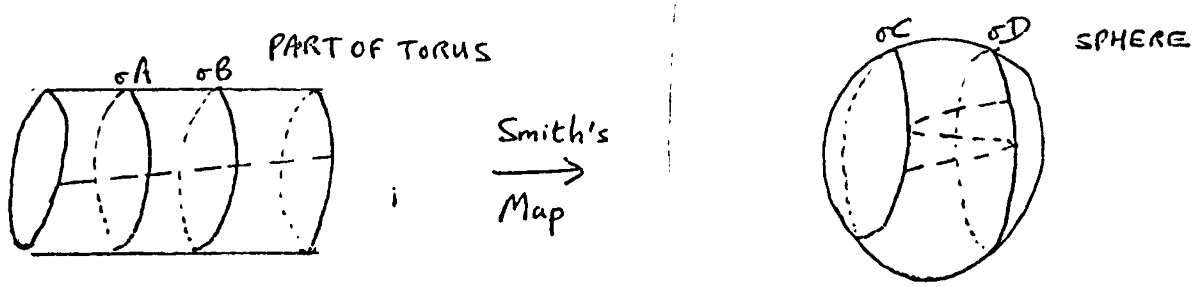
Applying the Gauss-Bonnet formula to the region $T = \tau(S^1\times[d,c])$ bounded by τC, τD

$$\int_T K*1 = -\int_{\tau C} k\,ds - \int_{\tau D} k\,ds$$

Suppose f is harmonic. By the maximum principle (1.2.12), τC cannot be concave towards T, otherwise if $p \in \sigma A$, a neighbourhood of p would map entirely to the concave side of T, contradicting the maximum principle (1.2.12). Thus $k \leq 0$ on τC. Similarly $k \leq 0$ on τD. Therefore $\int_T K*1 \geq 0$. But K is strictly negative on a dense subset of N, therefore $\int_T K*1 < 0$. Thus we have a contradiction, and therefore f cannot be harmonic. ∎

Remarks 4.4.3

This result is not true if N has positive curvature; for example, Smith's map of the torus to the sphere [Sm p. 95] exhibits pairs of opposite folds of the type prohibited by the theorem.



Our second result deals with a harmonic map $f:M \to N$ which restricts to a map $f:(H,\partial H) \to (K,\partial K)$ with a single fold. We shall see that the folding cannot occur "past" a closed geodesic around K.

Theorem 4.4.4

Let $f:M \to N$ be a smooth map which restricts to a map of handles $f:(H,\partial H) \to (K,\partial K)$. Suppose the curvature of N is strictly negative on a dense subset of N and suppose f is such that there exist C^1-diffeomorphisms, C^1 up to the boundary:

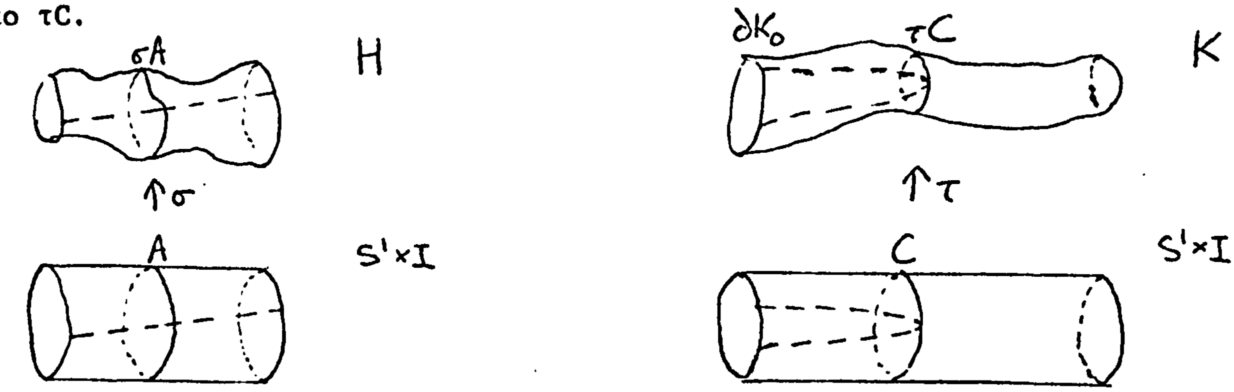
$$\sigma: (S^1\times I, S^1\times\partial I) \to (H,\partial H), \quad \tau: (S^1\times I, S^1\times\partial I) \to (K,\partial K)$$

such that, writing $\partial K_0 = \tau(S^1\times\{0\})$,

(1) $f(\partial H) \subset \partial K_0$, i.e. both ends of H map to one end of K,

(2) the singular set of $f|_H$ is a closed fold line σA, where $A = S^1\times\{a\}$, $0 < a < 1$, with image τC, where $C = S^1\times\{c\}$, $0 < c < 1$.

Then if f is harmonic, int f(H) cannot contain a closed geodesic homotopic to τC.



Proof

Suppose int f(H) does contain a closed geodesic Γ homotopic to τC. Then let T be the region of K bounded by Γ and τC: applying the

Gauss-Bonnet formula to T:

$$\int_T K*1 = - \int_{\tau C} k \, ds - \int_\Gamma k \, ds$$

As in (4.4.2), τC cannot be concave towards T and thus $k \leq 0$. On Γ, $k = 0$. Therefore $\int_T K*1 \geq 0$, contradicting the strictly negative curvature. ☐

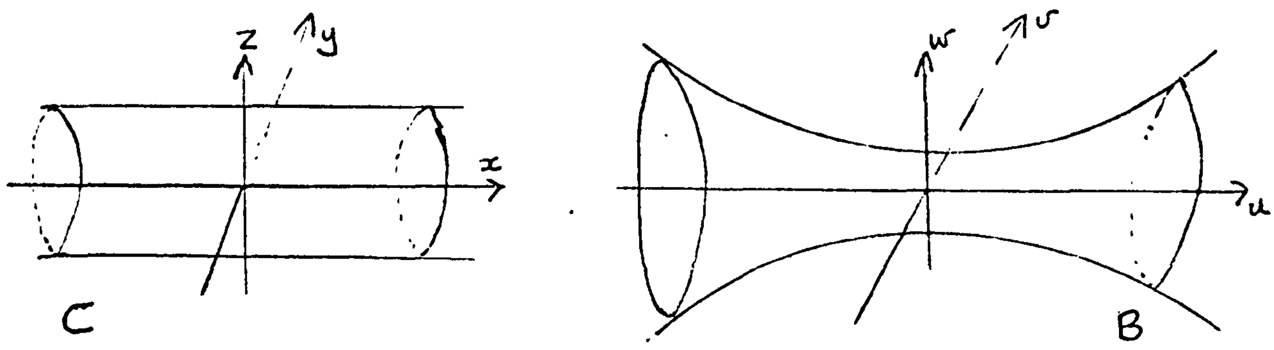
Remarks 4.4.5

Again this theorem is not true if N has positive curvature, Smith's map of the torus to the sphere (cf. (4.4.3)) providing an example. ☐

B Examples

Example 4.4.6

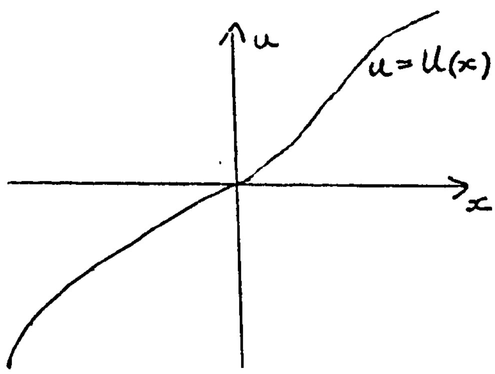
Let M = infinite circular cylinder C: $y^2 + z^2 = 1$, $-\infty < x < \infty$ in $\mathbb{R}^3$; let N = hyperboloid of revolution B: $v^2 + w^2 = 1 + u^2$ with metric induced from that of $\mathbb{R}^3$.



Note that B has strictly negative curvature except on the central geodesic $u = 0$, $v^2 + w^2 = 1$, where it is zero. Parametrise the cylinder by cylindrical coordinates (x,ϕ) where $\tan\phi = \frac{z}{y}$, and parametrise the hyperboloid similarly by (u,ψ) where $\tan\psi = \frac{w}{v}$. Suppose we are looking for "axially symmetric" harmonic mappings $f: C \to B$ of the form $u = U(x)$, $\psi = \phi$, where U is a smooth function on $(-\infty,\infty)$ such that $U(x) \to \infty$ as $x \to \infty$, $U(x) \to -\infty$ as $x \to -\infty$. We show U must be monotonic increasing with $U'(x) > 0 \ \forall\, x \in (-\infty,\infty)$. For if not let $U'(x)$ be zero for $x = x_o$. Choose x_o to be the least such zero. Then by our classification of singularities (2.2.11), $x = x_o$ must be a general fold which by (3.3.3) must contain a fold line. By axial symmetry the whole general fold must be a fold line. By (2.1.6, Notes (3)), U' has opposite signs on opposite sides of the fold line, and since $U(x) \to -\infty$ as $x \to -\infty$, U' changes sign from positive to negative as x increases through x_o. Since $U(x) \to \infty$ as $x \to \infty$, U' must change sign again; in fact let the next zero of U' bigger than x_o be x_1; then $x = x_1$ is a fold line with $U'(x)\begin{Bmatrix} < \\ > \end{Bmatrix} 0 \text{ for } \begin{Bmatrix} x_o < x < x_1 \\ x_1 < x < \text{some } x_2 \end{Bmatrix}$

We thus have double folding of the type discussed in (4.4.2). By this theorem, no such double folding can occur.

Therefore we conclude that for a harmonic map $f:C \to B$ of the form $u = U(x)$, $\psi = \phi$ with $U(x) \to \infty$ as $x \to \infty$, $U(x) \to -\infty$ as $x \to -\infty$, we must have U monotonic increasing with $U'(x) > 0 \ \forall\, x \in (-\infty,\infty)$.



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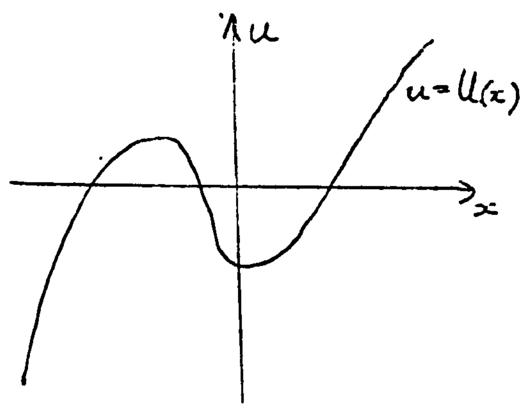

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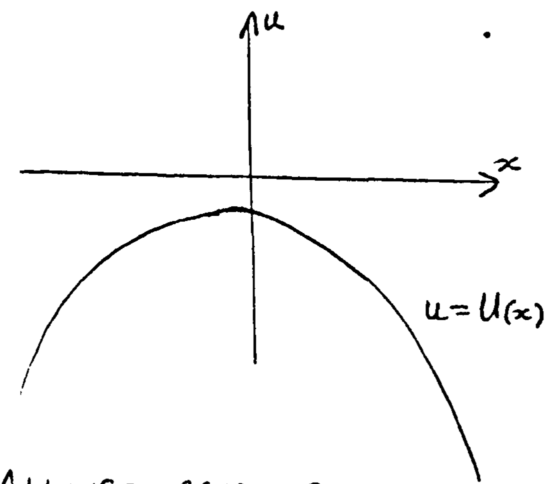
<u>Example 4.4.7</u>

With C, B as above we now look for harmonic maps of the form: $u = U(x)$, $\psi = \phi$ with $U(x) \to -\infty$ as $x \to \pm\infty$. This time there must exist at least one x_o such that $U'(x) = 0$ for $x = x_o$. As above this must be a fold line. We claim this is the only fold line. For if there is another, $x = x_1$, then we have double folding of the type in (4.4.2). Again this cannot occur. Thus $x = x_o$ is the only fold line.

We claim further that $U(x) \leqslant 0$ for all $x \in (-\infty,\infty)$. For suppose not. Then the image of C will contain the geodesic $u = 0$ which is prohibited by Theorem (4.4.4). Thus we have proved:

for a harmonic map $f:M \to N$ of the form $u = U(x)$, $\psi = \phi$ with $u \to -\infty$ as $x \to \pm\infty$, $U'(x)$ has precisely one zero which gives a maximum value for $u = U(x) \leqslant 0$.



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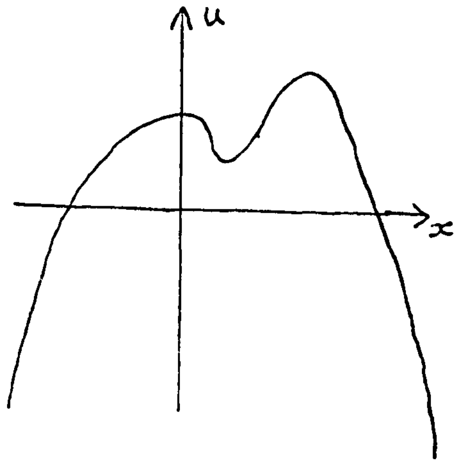

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5. CONSTRUCTION OF HARMONIC MAPS

5.1 Holomorphic maps into the 2-sphere

A Functions and Differentials

Let S be the Riemann sphere and let M be a compact Riemann surface of genus g. Many examples of harmonic maps $M \to S$ are provided by holomorphic maps; we here discuss how to construct such maps using the Riemann-Roch Theorem. We in fact construct meromorphic functions $M \to \mathbb{C} \cup \infty$ and then use the standard identification of $\mathbb{C} \cup \infty$ and S. We shall follow Springer [Sp].

Let M be equipped with local complex coordinates $z = x + iy$ centred on an arbitrary point $p \in M$. We first consider complex-valued functions f defined on a domain D of M. Such a function is harmonic if $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ throughout D (cf. 1.3.6).

Definitions 5.1.1 [Sp]

An analytic function f is a complex-valued function on D given locally by a Laurent series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{-n} z^{-n}$$

If at some $p \in M$, w.r.t. a complex coordinate z centred on p, this is of the form

$$(5.1A) \qquad f(z) = a_n z^n + a_{n+1} z^{n+1} + \dots \qquad (a_n \neq 0)$$

then if $n > 0$, we say f has a zero of multiplicity n (†) at p;
and if $n < 0$, we say f has a pole of multiplicity $-n$ at p. ∎
The multiplicity of a pole or zero is invariant under change of local complex coordinates [Sp p.173]. A pole or zero of multiplicity 1 is called a simple pole or zero.

Definitions 5.1.2 [Sp]

An analytic function f is called meromorphic if at each point p of its domain it has an expansion (5.1A); equivalently at the points where f

(†) We use "multiplicity" in preference to "order" for reasons apparent later is infinite (*), it has a pole. An analytic function is called holomorphic if at each point p of its domain it has an expansion (5.1A) with $n \geq 0$; equivalently, f is never infinite. ∎

The analytic functions, meromorphic functions, holomorphic functions form vector spaces over $\mathbb{C}$.

Let $C^{\infty}(L(TM,\mathbb{C}))$ denote the vector space of smooth complex-valued 1-forms (or "differentials") on M. Any 1-form can be written locally as $p\,dx + q\,dy$ where $p(x,y)$, $q(x,y)$ are smooth complex-valued functions.

Given a smooth complex-valued function defined on some domain of M, its derivative (or "total differential") df is the (smooth complex-valued) 1-form given locally by:

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Given any local complex coordinates (x,y), a coordinate disc is the set of points $(x - x_0)^2 + (y - y_0)^2 < r^2$.

Definitions 5.1.3 [Sp]

A 1-form ψ is said to be analytic, meromorphic, holomorphic, harmonic if, in each coordinate disc U, it is the total differential of some analytic, meromorphic, holomorphic, harmonic function f defined on U; if $\psi = df$ is a meromorphic differential expressed as the total differential of the meromorphic function f, ψ is said to have a pole (resp. zero) of multiplicity n at $p \in U$ according as f has a pole (resp. zero) of multiplicity n at p. ∎

Notes 5.1.4

(1) If ψ is a meromorphic differential with $\psi = df$, f meromorphic, then from the Cauchy-Riemann equations, it is easily seen that we can write:

$$\psi = f'(z)\,dz, \text{ where } dz = dx + i\,dy \text{ and } f'(z) = \frac{df}{dz}.$$

(2) Springer calls holomorphic 1-forms "abelian differentials".

(3) The definition of harmonic 1-form accords with a more general definition given by Eells-Sampson [E-S §3].

(*) We avoid using the word "singular" to avoid confusion with earlier usage.

(4) The analytic, meromorphic, holomorphic, harmonic 1-forms form vector spaces over $\mathbb{C}$. In particular, the holomorphic 1-forms on M form a vector space of dimension g over $\mathbb{C}$ [Sp 10.3].∎

Now we have an operation $*: C^{\infty}(L(TM,\mathbb{C})) \to C^{\infty}(L(TM,\mathbb{C}))$ called conjugation defined locally as follows:

(5.1B) if $w = p\,dx + q\,dy$ then $*w = -q\,dx + p\,dy$

Further if $w = p\,dx + q\,dy$ is a smooth 1-form, the exterior derivative dw is defined locally by:

$$dw = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}$$

In this notation f is harmonic iff $d{*}df = 0$. It can also be shown that w is a harmonic 1-form iff $dw = 0$, $d{*}w = 0$.

Proposition 5.1.5 [Sp]

If w is a harmonic 1-form then

(1) $*w$ is a harmonic 1-form,

(2) $w + i{*}w$ is a holomorphic 1-form.

If w is real-valued it is thus the real part of the holomorphic 1-form $w + i{*}w$.∎

B Divisors and Riemann-Roch Theorem

Let $P_1, \ldots, P_n$ be points of M and let $\alpha_1, \ldots, \alpha_n$ be integers. A divisor is a symbol $a = P_1^{\alpha_1} \ldots P_n^{\alpha_n}$. The degree $d[a]$ of a divisor is defined by $d[a] = \alpha_1 + \ldots + \alpha_n$. The divisor of a meromorphic function which has zeros at $P_1, \ldots, P_m$ of multiplicity $\alpha_1, \ldots, \alpha_m$ and poles at $P_{m+1}, \ldots, P_n$ of multiplicity $-\alpha_{m+1}, \ldots, -\alpha_n$ (and no other zeros or poles) is defined to be $P_1^{\alpha_1} \ldots P_m^{\alpha_m} P_{m+1}^{\alpha_{m+1}} \ldots P_n^{\alpha_n}$. Any two divisors can be multiplied or divided in an obvious way. The inverse of the divisor $a = P_1^{\alpha_1} \ldots P_n^{\alpha_n}$ is the divisor $a^{-1} = P_1^{-\alpha_1} \ldots P_n^{-\alpha_n}$. A divisor is called integral if $\alpha_k \geq 0\ \forall k$; if the quotient a/b of two divisors is an integral divisor we say b divides a or <u>a is a multiple of b</u>. Consider the space of meromorphic functions whose divisors are a multiple of a. If $a = P_1^{\alpha_1} \ldots P_m^{\alpha_m} P_{m+1}^{-\alpha_{m+1}} \ldots P_n^{-\alpha_n}$ (all $\alpha_i > 0$) we are thus considering the space of meromorphic functions such that $P_1, \ldots P_m$ are zeros of multiplicity $\geq \alpha_1, \ldots, \alpha_m$ and $P_{m+1}, \ldots P_n$ are possibly poles, with multiplicity $\leq \alpha_{m+1}, \ldots, \alpha_n$; additional zeros are allowed, but not additional poles. This space is a complex vector space. Let it have (complex) dimension $r[a]$. Similarly consider the space of holomorphic 1-forms whose divisors are a multiple of a. This space is a complex vector space of (complex) dimension, say, $i[a]$.

<u>Theorem 5.1.6 (Riemann Roch) [Sp th 10.10]</u>

Let M be a compact Riemann surface of genus g. Given a divisor a of degree $d[a]$ let $r[a^{-1}]$ denote the dimension of the vector space over $\underline{\mathbb{C}}$ of meromorphic functions whose divisors are multiples of a^{-1}, and let $i[a]$ denote the dimension of the vector space over $\underline{\mathbb{C}}$ of holomorphic 1-forms whose divisors are multiples of a. Then:

$$r[a^{-1}] = d[a] + i[a] - g + 1 \tag{5.1C}$$ ∎

<u>C Applications of the Riemann-Roch Theorem</u>

Again let M be a compact Riemann surface of genus g and let S be the Riemann sphere. We have the standard identification $i: S \to \underline{\mathbb{C}} \cup \infty$ given by stereographic projection (or by any meromorphic function $i: S \to \underline{\mathbb{C}} \cup \infty$ with a single simple pole). Composing with i^{-1}, a meromorphic function f on M gives rise to a holomorphic map $\hat{f}: M \to S$ into the Riemann sphere, viz: $\hat{f} = i^{-1} \circ f$. Conversely any holomorphic map $\hat{f}: M \to S$ arises from a meromorphic function f on M, $f = i \circ \hat{f}$. Now a holomorphic map is a ramified covering (3.2.3). The following result locates some of its branch points.

<u>Proposition 5.1.7</u>

A zero of f of multiplicity n ($n > 0$) is a point of $\hat{f}: M \to S$ of multiplicity n (2.1.21).

A pole of f of multiplicity n ($n > 0$) is a point of $\hat{f}:M \to S$ of multiplicity n.

(The converse is false.)

(Recall that, if $n > 1$, a point of multiplicity n of $\hat{f}:M \to S$ is a branch point of order $n - 1$).

Proof

For if f has a zero of multiplicity n at $p \in M$, in complex coordinates z centred on p, if $w \in \underline{\mathbf{C}}$, f has the form:

$$w = a_n z^n + a_{n+1} z^{n+1} + \dots \qquad (a_n \neq 0)$$

It is standard that there exists a complex analytic change of coordinates $z \to z'$ such that in the new coordinates f has the form:

$$w = z'^n$$

This is the canonical form for a point of multiplicity n (2.1.22). We can similarly show that if f has a pole of multiplicity n at p, then p is a point of multiplicity n for $\hat{f}$. ■

Proposition 5.1.8 (degree of f)

The degree of $\hat{f}:M \to S$ = the number of zeros of f up to multiplicity = the number of poles of f up to multiplicity.

Proof

By (3.2.5), the degree of $\hat{f}:M \to S$ = the number of inverse images of any point p of S counting each inverse image according to multiplicity. Taking $p = i^{-1}(0)$ or $i^{-1}(\infty)$ where $i: S \to \underline{\mathbf{C}} \cup \infty$ is the identification map gives the result.

Definition 5.1.9

A Weierstrass point is a point $P \in M$ where $i[P^g] > 0$. Equivalently, by the Riemann-Roch Theorem, a Weierstrass point is a point P where $r[P^{-g}] >$ i.e. there exist non-constant meromorphic functions with a pole at P of multiplicity $\leq g$ and no other poles.

Theorem 5.1.10 [Sp]

There are only a finite number of Weierstrass points; indeed if $g = 0$ or 1 there are none, if $g = 2$ there are precisely six, if $g > 2$ there are between $2g + 2$ and $(g - 1)g(g + 1)$ Weierstrass points. ∎

Lemma 5.1.11

If P is not a Weierstrass point, then for every $n \geq g + 1$, there exists a meromorphic function whose only pole is a pole of multiplicity n at P.

Proof

Since P is not a Weierstrass point, $i[P^g] = 0$. Thus $i[P^n] = 0$ for $n \geq g$ and thus from the Riemann-Roch Theorem:

$$r[P^{-n}] = n + 0 - g + 1 \quad (n \geq g)$$

Thus we get successively

$$r[P^{-g}] = 1 \qquad r[P^{-(g+1)}] = 2$$
$$r[P^{-(g+2)}] = 3 \qquad \ldots\ldots\ldots\ldots\ldots$$
$$r[P^{-(n-1)}] = n - g \qquad r[P^{-n}] = n - g + 1$$

Now $r[P^{-(n-1)}]$ denotes the dimension of the vector space of meromorphic functions with a pole at P of multiplicity at most $n - 1$ and no other poles. $r[P^{-n}]$ denotes the dimension of the vector space of meromorphic functions with a pole at P of multiplicity at most n and no other poles. The relation: $r[P^{-n}] = r[P^{-(n-1)}] + 1$ thus tells us that there is a (1-dimensional sub-space of) meromorphic function(s) with a pole at P of multiplicity precisely n and no other poles. ∎

Lemma 5.1.12

If $g > 1$ and P is a Weierstrass point, then there exists at least one n, $2 \leq n \leq g$, such that there is a meromorphic function whose only pole is a pole of multiplicity n at P.

Proof

Consider the sequence $r[1]$, $r[P^{-1}], \ldots, r[P^{-g}]$. Now $r[1] = 1$, since there are no non-constant meromorphic functions on M with no poles. Also $r[P^{-1}] = 1$, since there are no non-constant meromorphic functions on M with a simple pole at P, for such would by (5.18) give rise to a ramified cover $M \to S$ of degree 1 which by (3.2.5) would be a homeomorphism - a contradiction.

Further $r[P^{-g}] > 1$ by hypothesis. Thus, for at least one n $r[P^{-n}] > r[P^{-(n-1)}]$, which tells us there is at least one number n, $2 \leqslant n \leqslant g$, such that there exists a meromorphic function with a pole of multiplicity precisely n at P.∎

Theorem 5.1.13

Let M be a compact surface of genus g. There exist holomorphic maps $M \to S$ of degree n for:

all $n \geqslant 1$ if $g = 0$;

all $n \geqslant 2$ if $g = 1$ or 2;

all $n \geqslant g + 1$ and at least one n, $2 \leqslant n \leqslant g$, if $g \geqslant 3$.

Proof

Firstly, whatever the value of g, by (5.1.11), choosing any $P \in M$ which is not a Weierstrass point, there is a meromorphic function on M whose only pole is a pole of multiplicity n at P for all $n \geqslant g + 1$; by (5.1.8) this gives rise to a holomorphic function $M \to S$ of degree n. Additionally, if $g \geqslant 2$, choose $P \in M$ to be a Weierstrass point; by (5.1.12) there exists a meromorphic function whose only pole is a pole of multiplicity n at P for some n, $2 \leqslant n \leqslant g$. By (5.1.8) this gives rise to a holomorphic function $M \to S$ of degree n, $2 \leqslant n \leqslant g$. For $g = 2$, n must be 2.∎

Remarks

See §3.2D.∎

5.2 Harmonic Maps into the Flat Torus

A Construction of Harmonic Maps into the Torus

We shall study harmonic maps from a compact Riemann surface M of genus $g > 0$ into the flat torus $T = S' \times S'$ with a view to proving the theorem (5.2.14) stated at the end of this section on the maximum number of general folds, etc. for such a harmonic map.

We first show how to construct a harmonic map $f:M \to T$ in every homotopy class of maps. We follow Eells-Sampson [E-S p.128]. Terminology is as in the last section.

(1) Given a continuous map $u;M \to S'$ we can assign a cohomology class $[u] \in H^1(M,\underline{Z})$ viz: $[u] = u^*\{\text{generator of } H^1(S',\underline{Z})\}$. It is well-known that this assignment gives a bijection:

(5.2A) $\{\text{homotopy classes of maps } M \to S'\} \xrightarrow{h} H^1(M,\underline{Z})$

(2) Let w be a (real-valued) harmonic 1-form on M, let γ be a C^∞ closed curve. Then the period of w around γ is the number: $\int_\gamma w$. It can be shown to depend only on the homology class $[\gamma]$ of γ. We say w has integral periods if all the periods $\int_\gamma w$ are integers.

Let $[w]$ denote the cohomology class in $H^1(M,\underline{Z})$ such that $\langle [w],[\gamma]\rangle = \int_\gamma w$ where $\langle\ ,\ \rangle: H^1(M,\underline{Z}) \times H_1(M,\underline{Z}) \to \underline{Z}$ is the canonical pairing. By Hodge's Theorem, $w \to [w]$ defines a canonical bijection.

(5.2B) $\{\text{harmonic 1-forms with integral period}\} \xrightarrow{[\,]} H^1(M,\underline{Z})$

(Note that this is in fact an isomorphism of $\underline{Z}$-modules.)

(3) Thus, given a homotopy class of maps $M \to S'$, using the bijections (5.2A), (5.2B), we can construct a unique harmonic 1-form w on M with integral periods. We now "integrate" this 1-form to get a map $f:M \to S'$. Specifically, choose a base point p_0 for M; for any point $p \in M$, define:

$u(p) = \int_{\gamma_p} w$, where γ_p is a smooth path from p_0 to p. Different paths may give different numbers $u(p)$, but since the periods of w are integral, ~~f(p)~~ u(p) is well-defined modulo 1 and thus gives a well-defined map $u: M \to \mathbb{R}/\underline{\mathbb{Z}} = S'$. On any coordinate disc of M, $du = w$. Since $d{*}du = d{*}w = 0$ (see §5.1A), $\overset{u}{\cancel{f}}$ is <u>harmonic</u>.

(4) Thus in every homotopy class of maps $M \to S'$ we have constructed a harmonic map $u: M \to S'$. Now by taking a product of two such maps, in every homotopy class $M \to S'\times S'$ we can construct a harmonic map $f: M \to S'\times S'$ $p \to (u(p), v(p))$. $u, v: M \to S'$ have total differentials du, dv which are harmonic 1-forms with integral periods whose cohomology classes $[du]$, $[dv]$ correspond to the given homotopy class of maps $M \to S'\times S'$, viz:

$$\left\{\begin{matrix}\text{Homotopy classes}\\ \text{of maps } M \to S'\times S'\end{matrix}\right\} \xleftrightarrow[\text{isomorphism}]{\text{natural}} \left\{\begin{matrix}\text{Homotopy classes}\\ \text{of maps } M \to S'\end{matrix}\right\} \times \left\{\begin{matrix}\text{Homotopy classes}\\ \text{of maps } M \to S'\end{matrix}\right\}$$

$$\xleftrightarrow{h\times h} \underset{[du]}{H^1(M,\mathbb{Z})} \times \underset{[dv]}{H^1(M,\mathbb{Z})} \xleftrightarrow{[\;]\times[\;]} \left\{\begin{matrix}\text{pairs of harmonic 1-forms with}\\ \text{integral periods } du,\ dv\end{matrix}\right\}$$

$$\underset{\xleftarrow[\text{differentiate}]{}}{\xrightarrow{\text{integrate}}} \left\{\begin{matrix}\text{Harmonic maps } f: M \to S'\times S'\\ p \mapsto (u(p), v(p))\end{matrix}\right\}$$

<u>B Singularities of f</u>

Let $f: M \to S'\times S'$ be a harmonic map. We use the notation in (4) above. (u,v) will also denote a point of $S'\times S'$. We study the singularities of f. Our starting point is:

<u>Lemma 5.2.1</u>

p is a singular point of $f: M \to S'\times S'$ $p \to (u(p), v(p))$ if and only if $\lambda\, du(p) + \mu\, dv(p) = 0$ for some $\lambda, \mu \in \mathbb{R}$ not both zero.

<u>Proof</u>

If we take smooth local coordinates (x,y) near p, df has matrix form:

$$(5.2C)\quad \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} = \begin{bmatrix} du \\ dv \end{bmatrix} \quad \text{where } du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right),\ dv = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

Thus p is a singular point $\Leftrightarrow$ the matrix $\begin{bmatrix} du(p) \\ dv(p) \end{bmatrix}$ is singular $\Leftrightarrow \lambda\, du(p) + \mu\, dv(p) = 0$ for some $\lambda, \mu \in \mathbb{R}$ not both zero ∎

<u>Corollary 5.2.2</u>

The singular set of f includes the zeros of du, dv. ∎

We now use (5.2.1) to study the singular set of f. We distinguish four different cases depending on the relationship between the harmonic 1-forms du, dv.

Proposition 5.2.2 (Case 1)

If $du = dv = 0$ then $f: M \to S^1 \times S^1$ is constant.

Proof

For if $du = dv = 0$, then u and v are constant, therefore f is constant. ∎

Now we can consider the set of harmonic 1-forms on M as a module over either $\mathbb{R}$ or $\mathbb{Z}$. We say two harmonic 1-forms w, w' are linearly dependent over $\mathbb{R}$ iff there exist $\lambda, \mu \in \mathbb{R}$ not both zero such that $\lambda w + \mu w' = 0$; we say w, w' are linearly dependent over $\mathbb{Z}$ iff there exist $\lambda, \mu \in \mathbb{Z}$ not both zero such that $\lambda w + \mu w' = 0$.

Proposition 5.2.3 (Case 2)

If du, dv are not both zero but are linearly dependent over $\mathbb{Z}$, then f maps M onto a closed geodesic of $S^1 \times S^1$.

Proof

There exist $\lambda, \mu \in \mathbb{Z}$ not both zero such that $\lambda du + \mu dv = 0$. By (5.2.1) all points of M are thus singular points; by (3.1.4) f maps M onto a closed geodesic (in fact the geodesic is $\lambda u + \mu v =$ constant which winds round the torus μ times in the "u-direction" for every $-\lambda$ times in the "v-direction".) ∎

Case 3

Lemma 5.2.4

Suppose w, w' are harmonic 1-forms with integral periods. Then w, w' are linearly dependent over $\mathbb{R} \Leftrightarrow$ w, w' are linearly dependent over $\mathbb{Z}$.

Proof

$\Leftarrow$ trivial.

$\Rightarrow$ If w, w' are linearly dependent over $\mathbb{R}$, then:

$$\lambda w + \mu w' = 0 \quad (\lambda, \mu \in \mathbb{R}, \text{ not both zero})$$

Without loss of generality assume $\lambda \neq 0$; then $w = -\frac{\mu}{\lambda}w'$. It follows that the period of w around the closed path $\gamma = -\frac{\mu}{\lambda}$ (period of w' around γ). But the periods of w, w' are <u>integral</u>, therefore $\frac{\mu}{\lambda}$ is <u>rational</u>, therefore $\frac{\mu}{\lambda} = \frac{\tilde{\mu}}{\tilde{\lambda}}$ where $\tilde{\mu}$, $\tilde{\lambda}$ are <u>integral</u> and then $\tilde{\lambda}w + \tilde{\mu}w' = 0$ ($\tilde{\lambda}, \tilde{\mu} \in \mathbb{Z}$, not both zero), showing w, w' are linearly dependent over $\mathbb{Z}$. ∎

Now consider the real-valued harmonic 1-forms with integral periods: du, dv. As in §5.1A we can form the (real-valued) <u>conjugate</u> harmonic 1-forms *du, *dv and the (complex-valued) holomorphic 1-forms:

(5.2D) $\quad \theta = du + i{*}du \qquad \phi = dv + i{*}dv$

By (5.1.5) du, dv are the <u>real parts</u> of θ, ϕ. As remarked in (5.1.4) the set of holomorphic 1-forms on M forms a vector space over $\mathbb{C}$ (of dimension equal to the genus of M); we say θ, ϕ are <u>linearly dependent over $\mathbb{C}$</u> iff there exist $\alpha, \beta \in \mathbb{C}$ not both zero such that $\alpha\theta + \beta\phi = 0$. We are now ready for

<u>Proposition 5.2.5 (Case 3)</u>

If <u>du, dv are linearly independent over $\mathbb{Z}$</u> and <u>θ, ϕ are linearly dependent over $\mathbb{C}$</u>, then $f: M \to S' \times S'$ is a ramified covering of ramification index $2g - 2$ where $g = \text{genus}(M)$.

<u>Proof</u>

First note, by (5.2.4), du, dv are linearly independent <u>over $\mathbb{R}$</u>. Now θ, ϕ are linearly dependent over $\mathbb{C}$, thus there exist $\alpha, \beta \in \mathbb{C}$, not both zero, such that:

$$\alpha\theta + \beta\phi = 0$$

From the definition of θ, ϕ (5.2D):

(5.2E) $\quad \alpha(du + i{*}du) + \beta(dv + i{*}dv) = 0$

Taking complex conjugates:

(5.2F) $\quad \bar{\alpha}(du - i{*}du) + \bar{\beta}(dv - i{*}dv) = 0$

Without loss of generality assume $\beta \neq 0$. Forming $\bar{\beta}$(5.2E) + β(5.2F) eliminates $*dv$ between (5.2E), (5.2F), giving:

(5.2G) $\quad dv = \lambda du + \mu * du$

where $\lambda = -\dfrac{\alpha\bar{\beta} + \bar{\alpha}\beta}{2\beta\bar{\beta}} \qquad \mu = -\dfrac{i(\alpha\bar{\beta} - \bar{\alpha}\beta)}{2\beta\bar{\beta}}$

We see that λ, μ are <u>real</u>, and, since du, dv are linearly independent over $\mathbb{R}$, $\mu \neq 0$.

Now, taking local complex coordinates (x,y) for M, the derivative of f may be written in matrix form:

$$df = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial x} \end{bmatrix} = \begin{bmatrix} du \\ dv \end{bmatrix} \quad \text{where} \quad du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), dv = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right).$$

Thus for the Jacobian J of f,

$$J = \det \begin{bmatrix} du \\ dv \end{bmatrix} = \det \begin{bmatrix} du \\ \lambda du + \mu * du \end{bmatrix} \text{ (by (5.2G))} = \mu \det \begin{bmatrix} du \\ * du \end{bmatrix} \text{ (by row operations)}$$

$$= \mu \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} \end{vmatrix} = \mu\left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\}$$

Since $\mu \neq 0$ the Jacobian of f is zero if and only if $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are zero, i.e. $p \in M$ is a singularity of f if and only if p is a zero of du. Now any harmonic 1-form has $2g - 2$ zeros up to multiplicity [Sp], in particular these zeros are isolated. Thus f has an isolated singularity at each zero of du - by (2.2.11) this must be a branch point - and f has no other singularities. Hence f is a ramified covering; by Hurwitz' formula (3.2.7) it has ramification index $2g - 2$. ∎

<u>Remark</u>

It can easily be seen by expanding u, v in local coordinates that if du has a zero of multiplicity n at a point $p \in M$, then f has a branch point of order n at this point. ∎

<u>Case 4</u>

<u>Proposition 5.2.6 (Case 4)</u>

If <u>du, dv are linearly independent over $\mathbb{Z}$ and θ, ϕ are linearly</u>

<u>independent over $\mathbb{C}$</u>, then the harmonic map $f:M \to S^1 \times S^1$ has singular set Σ consisting of at most $2g - 2$ branch points, $(2g - 2)(6g - 6)$ general folds and $6g - 6$ meeting points of general folds, where $g = \text{genus}(M)$.

<u>Proof</u>

Let z be a local complex coordinate for M centred on some point $p \in M$. Then since $f:M \to S^1 \times S^1$ is harmonic, its components $u,v:M \to S^1$ are locally the real part of holomorphic functions viz:

$$\text{(5.2H)} \qquad \begin{aligned} u &= \mathrm{re}(a_\ell z^\ell + a_{\ell+1} z^{\ell+1} + \ldots) \\ v &= \mathrm{re}(b_k z^k + b_{k+1} z^{k+1} + \ldots) \end{aligned} \qquad a_j, b_j \in \mathbb{C},\ \begin{aligned} &a_\ell \neq 0,\ \ell \geq 1, \\ &b_k \neq 0,\ k \geq 1 \end{aligned}$$

As before, du, dv are the real parts of the holomorphic differentials $\theta = du + i{*}du$, $\phi = dv + i{*}dv$; it is easily seen that θ, ϕ are given locally by:

$$\text{(5.2J)} \qquad \begin{aligned} \theta &= (\ell a_\ell z^{\ell-1} + (\ell+1) a_{\ell+1} z^\ell + \ldots)dz \\ \phi &= (k b_k z^{k-1} + (k+1) b_{k+1} z^k + \ldots)dz \end{aligned}$$

Thus θ, ϕ are of the form:

$$\begin{aligned} \theta &= (c_n z^n + c_{n+1} z^{n+1} + \ldots)dz \qquad && c_j \in \mathbb{C},\ c_n \neq 0,\ n \geq 0, \\ \phi &= (d_m z^m + d_{m+1} z^{m+1} + \ldots)dz \qquad && d_j \in \mathbb{C},\ d_m \neq 0,\ m \geq 0 \end{aligned}$$

We may form the <u>quotient</u> θ/ϕ defined locally by:

$$\theta/\phi = \frac{c_n z^n + c_{n+1} z^{n+1} + \ldots}{d_m z^m + d_{m+1} z^{m+1} + \ldots}$$

$$\text{(5.2K)} \qquad = \frac{c_n z^n + c_{n+1} z^{n+1} + \ldots}{d_m z^m} \left(1 + \frac{d_{m+1}}{d_m} z + \ldots\right)^{-1}$$

θ/ϕ is a meromorphic function on M [Sp 6.11]. Note that if $\theta(p)$, $\phi(p)$ are not both zero, $\theta/\phi(p) = \theta(p)/\phi(p)$. Further note that by the hypothesis of linear independence of θ and ϕ, θ/ϕ is not constant.

<u>Lemma 5.2.7</u>

Suppose θ has a zero of multiplicity n at p and ϕ has a zero of multiplicity m at p. Then if $n > m$, θ/ϕ has a zero of multiplicity $n-m$ at p; if $n < m$, θ/ϕ has a pole of multiplicity $m-n$ at p; if $n = m$

θ/ϕ has neither a zero nor a pole at p.

Proof

Directly from (5.2K). ∎

Corollary 5.2.8

The number of zeros of θ/ϕ up to multiplicity $\leq$ number of zeros of θ up to multiplicity.

Proof

Immediate from (5.2.7). ∎

We now investigate the value of θ/ϕ at a singular point of our harmonic map $f: M \to S' \times S'$. Recall the notation:

$$\Sigma_i = \{p \in M:\ \dim\ker df(p) = i\} \quad (i = 0,\ 1,\ 2),\ \Sigma = \Sigma_1 \cup \Sigma_2$$

Lemma 5.2.9

p is a singular point of f other than a branch point $\Leftrightarrow$ $\theta/\phi(p) \in \mathbb{R} \cup \infty$.

Proof

(a) Suppose $p \in \Sigma_1$. Then by (5.2.1), $\lambda du(p) + \mu dv(p) = 0$ for some $\lambda, \mu \in \mathbb{R}$ not both zero. Applying the operator $*$:

$$\lambda * du(p) + \mu * dv(p) = 0$$

Multiplying by i and adding to previous equation:

(5.2L) $\quad \lambda\theta(p) + \mu\phi(p) = 0$

Now since $p \in \Sigma_1$, $df(p) \neq 0$, therefore not both $du(p)$, $dv(p)$ are zero. It follows that not both $\theta(p)$, $\phi(p)$ are zero. As remarked previously it follows that $\theta/\phi(p) = \theta(p)/\phi(p)$. It now follows from (5.2L) that $\theta/\phi(p) = -\mu/\lambda$ and since λ, μ are real and not both zero, we conclude $\theta/\phi(p) \in \mathbb{R} \cup \infty$.

(b) Suppose $p \in \Sigma_2$ but p is not a branch point. Then (2.2.11) shows p must be the meeting point of general folds. We can choose a sequence of points $q_n \to p$ with each $q_n \in$ a general fold. Then each $q_n \in \Sigma_1$, and thus by (a) above $\theta/\phi(q_n) \in \mathbb{R} \cup \infty$. By continuity of θ/ϕ, it follows that

$\theta/\phi(p) \in \mathbb{R} \cup \infty$.

(c) Suppose p is a branch point. Then in local complex coordinate z centred on p, f has the form (5.2H). From our table of the singularities of a harmonic map in §2.2C, we see that $\ell = k$ = multiplicity of branch point, $\ell = k > 1$, and $\mathrm{im}(a_\ell \bar{b}_k) \neq 0$.

Thus (5.2J) becomes:

$$\theta = (ka_k z^{k-1} + (k+1)a_{k+1}z^k + \ldots)dz$$
$$\phi = (kb_k z^{k-1} + (k+1)b_{k+1}z^k + \ldots)dz$$

whence $\theta/\phi = \dfrac{(ka_k z^{k-1} + (k+1)a_{k+1}z^k + \ldots)}{kb_k z^{k-1}} \left(1 + \dfrac{(k+1)b_{k+1}}{kb_k} z + \ldots\right)^{-1}$

$$= \frac{a_k}{b_k} + \text{terms in z and higher order.}$$

Hence, $\theta/\phi(p) = \dfrac{a_k}{b_k}$. By the condition $\mathrm{im}(a_k \bar{b}_k) \neq 0$, $\dfrac{a_k}{b_k}$ is not real.

Thus $\theta/\phi(p) \notin \mathbb{R} \cup \infty$.

(d) Now suppose $\theta/\phi(p) \in \mathbb{R} \cup \infty$. In local complex coordinate z centred on p, suppose: $\theta = (c_o + c_1 z + \ldots)dz$

$$\phi = (d_o + d_1 z + \ldots)dz$$

Now if c_o, d_o are both zero, $\theta(p) = \phi(p) = 0$. It follows that $du(p) = dv(p) = 0$, hence p is a singular point. Otherwise, if c_o, d_o are not both zero, then $\theta/\phi(p) = c_o/d_o$. From the hypothesis that $\theta/\phi(p) \in \mathbb{R} \cup \infty$ we deduce $c_o/d_o \in \mathbb{R} \cup \infty$, therefore we can write $c_o/d_o = -\mu/\lambda$ where $\lambda, \mu \in \mathbb{R}$ are not both zero. Since $c_o = \theta(p)$, $d_o = \phi(p)$ it follows

$$\lambda\theta(p) + \mu\phi(p) = 0.$$

Taking real parts, $\lambda du(p) + \mu dv(p) = 0$. Therefore by (5.2.1), p is a singular point.

Putting (a), (b), (c), (d) together proves Lemma (5.2.9). ∎

We now use Lemma (5.2.9) to investigate the singular set of f. Recall the standard identification $S \to \mathbb{C} \cup \infty$ of the Riemann sphere with the extended complex plane. By the real circle of S we shall mean

$i^{-1}(\mathbb{R}\cup\infty)$, i.e. the circle of S corresponding to the extended real line. Using i we convert the meromorphic function θ/ϕ on M into a holomorphic mapping $\widehat{\theta/\phi}:M \to S$ defined by $\widehat{\theta/\phi} = i^{-1}\circ\theta/\phi$. Now as remarked previously θ/ϕ is non-constant, therefore $\widehat{\theta/\phi}$ is non-constant and therefore $\widehat{\theta/\phi}$ is a ramified covering whose only singularities are branch points (3.2.3).

<u>Lemma 5.2.10</u>

$\Sigma \setminus \{\text{branch points of } f\} = \widehat{\theta/\phi}^{-1}(\text{real circle of } S)$.

<u>Proof</u>

Immediate from (5.2.9) (Here, as usual, Σ denotes the singular set of f).∎

<u>Lemma 5.2.11</u>

$\widehat{\theta/\phi}:M \to S$ has degree d, $2 \leq d \leq 2g - 2$, where g = genus(M)

<u>Proof</u>

By (5.1.8), the degree of $\widehat{\theta/\phi}:M \to S$ = d = number of zeros of θ/ϕ up to multiplicity. (Note $d \geq 0$ since $\widehat{\theta/\phi}$ is holomorphic). By (5.2.8) $d \leq$ number of zeros of θ up to multiplicity. But any holomorphic 1-form has $2g - 2$ zeros up to multiplicity where g = genus(M). Therefore $d \leq 2g - 2$.

Further, $d \neq 0$ since $\widehat{\theta/\phi}$ is non-constant, and $d \neq 1$, otherwise, by (3.2.6), $\widehat{\theta/\phi}$ would be a homeomorphism from M to S, which is impossible. Therefore $2 \leq d \leq 2g - 2$.∎

<u>Corollary 5.2.12</u>

$\widehat{\theta/\phi}:M \to S$ has ramification index r, $2g + 2 \leq r \leq 6g - 6$.

<u>Proof</u>

By Hurwitz' formula (3.2.7), $r = 2d - (2 - 2g)$. Applying (5.2.11) gives inequality.∎

<u>Lemma 5.2.13</u>

$\widehat{\theta/\phi}^{-1}(\text{real circle of } S)$ consists of at most $(2g - 2)(6g - 6)$ disjoint analytic 1-submanifolds and at most $6g - 6$ meeting points of these 1-submanifolds (§2.1B).

<u>Proof</u>

(a) Write β = set of branch points of $\widehat{\theta/\phi}$. Suppose firstly $\widehat{\theta/\phi}:M \to S$ has no branch points mapping to the real circle. Then since $\widehat{\theta/\phi}:M\setminus\beta \to S$ is a smooth covering, it is clear that $\widehat{\theta/\phi}^{-1}$(real circle) is a certain number, s, of disjoint 1-submanifolds (homeomorphic to circles) where $1 \leq s \leq d$ = degree of $\widehat{\theta/\phi}$.

(b) Suppose, secondly, that $\widehat{\theta/\phi}:M \to S$ has t branch points mapping to a set α of s distinct points on the real circle $\rho \subset S$. By (5.2.12) $s \leq t \leq 6g - 6$. Then since $\widehat{\theta/\phi}:M\setminus\beta \to S\setminus\alpha$ is a smooth covering surface and since $\rho\setminus\alpha$ consists of s disjoint 1-submanifolds, then $\widehat{\theta/\phi}^{-1}(\rho\setminus\alpha)$ consists of sd disjoint 1-submanifolds.

<u>Example</u> $s = 3$
$d = 2$

Therefore $(\widehat{\theta/\phi})^{-1}$(real circle) consists of these sd disjoint 1-submanifolds together with the t branch points of $\widehat{\theta/\phi}$ which are easily seen to be meeting points of these 1-submanifolds. Using $s \leq t \leq 6g - 6$ and $d \leq 2g - 2$ and combining this result with (a) above proves the lemma. ∎

Combining (5.2.10), (5.2.13) we conclude $\Sigma\setminus$\{branch points of f\} consists of at most $(2g - 2)(6g - 6)$ disjoint 1-submanifolds and $6g - 6$ meeting points. By (3.1.7) the 1-submanifolds must be general folds, also by (3.1.7C) there are at most $2g - 2$ branch points of f; the proposition 5.2.6 now follows. ∎

<u>Summing Up of Cases 1 - 4</u>

<u>Theorem 5.2.14</u>

Let $f:M \to T$ be a harmonic map from a compact Riemann surface M of genus g into the flat torus $T = S^1 \times S^1$. Then one of the following holds:

(1) f is constant;

(2) f maps M onto a closed geodesic;

(3) f is a ramified covering with ramification index $2g - 2$;

(4) f has singular set Σ consisting of at most $2g - 2$ branch points, $(2g - 2)(6g - 6)$ general folds, and at most $(6g - 6)$ meeting points.

Proof

Combine (5.2.2),(5.2.3), (5.2.5), (5.2.6). ∎

C Special Case - genus M = 1

Corollary 5.2.15

Let $f:M_1 \to T$ be a harmonic map from any torus M_1 (i.e. any compact surface of genus 1) to the flat torus T. Then one of the following holds:

(1) f is constant;

(2) f maps M_1 onto a closed geodesic;

(3) f is a smooth covering.

Proof

Put $g = 1$ in Theorem (5.2.14). (Note: (5.2.11) shows that we cannot have case (4).) ∎

Remark

This result is given by Fuller [Fu]. ∎

D Special Case - genus M = 2

Proposition 5.2.16

In the case that $f:M_2 \to T$ is a harmonic map from a compact Riemann surface M_2 of genus 2 to the flat torus, then one of the following holds:

(1) f is constant;

(2) f maps M_2 to a closed geodesic;

(3) f is a ramified covering with ramification index 2;

(4) f has at most 12 general folds and at most 6 meeting points, these being Weierstrass points, there being no branch points.

Proof

We need only prove (4). This corresponds to case (4) in §5.2B. Thus $f:M_2 \to T$ is given by $p \to (u(p),v(p)) \in S'\times S'$ where du, dv are linearly independent over $\mathbb{R}$ and $\theta = du + i{*}du$, $\phi = dv + i{*}dv$ are linearly independent over $\mathbb{C}$.

(a) Suppose first p is a branch point of $f:M_2 \to T$. Then $df(p) = 0$. Therefore, $du(p) = dv(p) = 0$; therefore $\theta(p) = \phi(p) = 0$. Thus the two holomorphic 1-forms θ, ϕ vanish at p. But since θ, ϕ are linearly independent over $\mathbb{C}$, they span the 2-dimensional complex vector space of all holomorphic 1-forms. Since not all holomorphic 1-forms vanish at p [Sp] we have a contradiction. Thus f has no branch points.

(b) Suppose $p \in M_2$ is a meeting point of general folds for $f:M_2 \to T$. Then p is a branch point for the holomorphic mapping $\widehat{\theta/\phi}:M \to S$. Now this mapping has degree 2 (5.2.11), and thus the multiplicity of p must be precisely 2; hence no other point q maps to f(p). Now we compose with a conformal diffeomorphism $h:S \to S$ which maps f(p) to ∞. Then $h\circ(\widehat{\theta/\phi}):M \to S$ is a holomorphic mapping with a branch point of multiplicity 2 mapping to ∞ and no other points mapping to ∞. By (5.1.7) this defines a meromorphic map with a pole of multiplicity 2 at p and no other poles. In the notation of (5.1.6) $r[p^{-2}] > 1$. Thus, by definition (5.1.9), p is a Weierstrass point of M. ∎

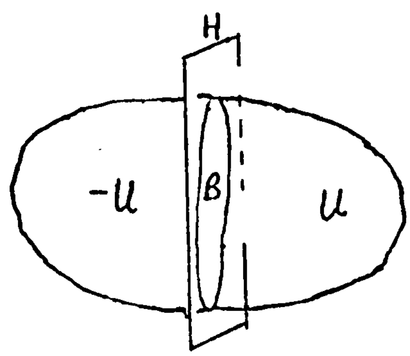
5.3 Reflection Laws

A Introduction

Let U be a domain (= connected open set) of the right-half space $x^1 > 0$ of $\mathbb{R}^m$ such that: (1) ∂U contains an open subset of the hyperplane H: $x^1 = 0$, (2) U contains the portion $x^1 > 0$ of a neighbourhood of each point of H. We say, following Lewy [Le 2], U is adjacent to H on the side $x^1 > 0$. Now let $B = \operatorname{int}(\bar{U} \cap H)$ and let $f: U \to \mathbb{R}$ be a harmonic function which has a continuous extension to $U \cup B$. Then the classical reflection principle asserts that if f vanishes on B then it has an extension to the other side of H by the reflection law:

$$f(-x^1, x^2, \ldots, x^m) = -f(x^1, x^2, \ldots, x^m), \quad (x^1, x^2, \ldots, x^m) \in U \subset \mathbb{R}^m$$

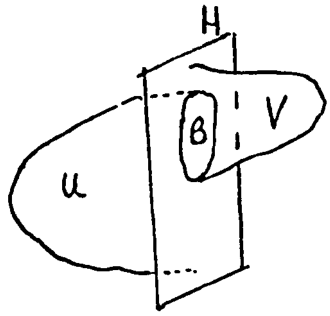
Note that f is now defined on the domain $U \cup (-U) \cup B$ where $-U = \{-x : x \in U\}$ and f is harmonic on this domain (see diagram below).



We shall give generalisations of this law to harmonic maps, and show how a reflection law might be applied to the construction of a harmonic map from the torus to the sphere (of even degree), given a suitable harmonic map from a square or disc to a hemisphere.

B Glueing Lemma

Let M, N be smooth Riemannian manifolds, of dimensions m, n respectively, and suppose we have two harmonic maps $f:U \to N$, $h:V \to N$ defined in coordinate domains adjacent to a hypersurface H and on opposite sides (see diagram).



Let $B = \mathrm{int}(\bar{U} \cap \bar{V} \cap H)$ be their common boundary.

Definition 5.3.1

We say that we can glue f and h together if we can construct a smooth map k defined on $W = U \cup V \cup B$ extending f and h. ∎ By continuity of the tension field such a map is necessarily harmonic.

Lemma 5.3.2 (Glueing Lemma)

f and h can be glued together if and only if:

(1) f and h have C^2 extensions to $U \cup B$, $V \cup B$ respectively - which we shall also denote by f, h (i.e. f, h are C^2 up to the common boundary);

(2) $f(p) = h(p)$ $\forall p \in B$ (i.e. f and h agree on B);

(3) $\frac{\partial f}{\partial n}(p) = \frac{\partial h}{\partial n}(p)$ for all $p \in B$ (i.e. the normal derivatives agree on B).

Proof

Necessity is evident; we prove sufficiency.

Hypothesis (2) ensures that f and h can be glued together to get a C^0 map $k:W \to N$. We must show that k is smooth at the common boundary B.

We show k is C^1:

Let $p \in B$. Take (smooth) coordinates $(x^1,\ldots,x^m)$ near $p \in B$ such that B is given by $x^1 = 0$, U by $x^1 < 0$ and V by $x^1 > 0$, and such that the x^1-axis is normal to B. Also take arbitrary (smooth) coordinates $(u^1,\ldots,u^n)$ for N near f(p).

$$\therefore \quad \frac{\partial f^\gamma}{\partial x^i}(p) = \frac{\partial h^\gamma}{\partial x^i}(p) \quad \forall p \in B \quad \forall \gamma = 1, \ldots, n, \quad \forall i = 2, \ldots, m$$

By hypothesis (3) this is also true for i = 1.

Thus k is C^1 at all points $p \in B$.

<u>We show k is C^2:</u>

Hypotheses (2) and (3) show:

$$\text{(5.3A)} \qquad \frac{\partial^2 f^\gamma}{\partial x^i \partial x^j}(p) = \frac{\partial^2 h^\gamma}{\partial x^i \partial x^j}(p) \quad \forall p \in B, \quad \forall \gamma, \quad \forall (i,j) \neq (1,1)$$

For the case (i,j) = (1,1), we write down the tension field equations for f and h at the point $p \in B$:

$$g^{ij}\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - g^{ij} f_k^\gamma \Gamma_{ij}{}^k + g^{ij} f_i^\alpha f_j^\beta L_{\alpha\beta}{}^\gamma = 0$$

$$g^{ij}\frac{\partial^2 h^\gamma}{\partial x^i \partial x^j} - g^{ij} h_k{}^\gamma \Gamma_{ij}{}^k + g^{ij} h_i{}^\alpha h_j{}^\beta L_{\alpha\beta}{}^\gamma = 0$$

Subtracting these, since:

$$\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} = \frac{\partial^2 h^\gamma}{\partial x^i \partial x^j} \quad \forall (i,j) \neq (1,1), \ \forall \gamma,$$

all terms cancel in pairs except the terms:

$$g^{11}\frac{\partial^2 f^\gamma}{(\partial x^1)^2}, \quad g^{11}\frac{\partial^2 h^\gamma}{(\partial x^1)^2}$$

We must therefore have:

$$\frac{\partial^2 f^\gamma}{(\partial x^1)^2} = \frac{\partial^2 h^\gamma}{(\partial x^1)^2} \quad \forall \gamma, \ \forall p \in B.$$

Together with (5.3A) this shows that k is C^2 at all points $p \in B$.

<u>We show k is C^∞ and harmonic:</u>

For we have shown that k is C^2 at all points of B, certainly k is C^∞ on $W \setminus B$ since it then equals f or h. Thus the tension field of k is continuous on W, and since it is zero on $W \setminus B$ (since f and h are harmonic), it must thus be zero on B. Thus k is harmonic and by (1.1.4) k is C^∞. ∎

Notes

The lemma uses the harmonic equations in an essential way and is clearly not true for non-harmonic maps f and h. ∎

C Isometric Reflection

Definition 5.3.3

We shall say that a Riemannian manifold M has isometric symmetry across a hypersurface (†) $H \subset M$ if there exists an isometry $r: M \to M$ of order 2 leaving H pointwise fixed.

The most obvious example of such an isometry is reflection across a hyperplane of $\mathbb{R}^m$. Note that if M is oriented, such an isometry is orientation reversing; further, recall a smooth submanifold of M is said to be totally geodesic if every geodesic of the submanifold is a geodesic of M.

Proposition 5.3.4

If M has isometric symmetry across the hypersurface H, H must be totally geodesic.

Proof

Let $p \in H$ and choose (smooth) coordinates $(x^1,\dots,x^m)$ centred on p such that H is given by $x^1 = 0$ and the x^1-axis is perpendicular to H at p. Firstly we show $\Gamma_{ij}^{\ 1}(p) = 0 \ \forall i,j > 1$.

For use the coordinates $(x^1,\dots,x^m)$ to calculate the Christoffel symbols $\Gamma_{ij}^{\ 1}(p)$, and then use new coordinates $(x'^1,\dots,x'^m)$ where $x'^1 = -x^1$, $x'^i = x^i$ $(i = 2,\dots,m)$ to calculate $\Gamma'_{ij}{}^{1}(p)$. By the transformation law for Christoffel symbols [Hi] , $\Gamma'_{ij}{}^{1}(p) = -\Gamma_{ij}^{\ 1}(p) \ \forall i,j>1$, and by symmetry $\Gamma'_{ij}{}^{1}(p) = \Gamma_{ij}^{\ 1}(p)$. Therefore $\Gamma_{ij}^{\ 1}(p) = 0 \ \forall i,j > 1$. We show that this implies H is totally geodesic.

For let $x^i = x^i(t)$ $(i = 1,\dots,m)$ be parametric equations for a

(†) A hypersurface is a smooth submanifold of codimension 1.

geodesic of H. Then $x^1 \equiv 0$ and $x^2,\ldots,x^m$ satisfy [Hi p.58]:

$$\frac{d^2x^k}{dt^2} + \Gamma_{ij}^{\;k}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \qquad (k = 2,\ldots,m) \tag{5.3C}$$

But by (5.3B):

$$\frac{d^2x^1}{dt^2} + \Gamma_{ij}^{\;1}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 + 0 = 0.$$

Therefore (5.3C) holds for $k = 1,\ldots,m$; therefore $x^i = x^i(t)$ $(i = 1,\ldots,m)$ is a geodesic in M. We have thus demonstrated that any geodesic in H is a geodesic in M. Thus by definition H is totally geodesic. ∎

Now the classical reflection principle mentioned in §5.3A uses the isometric symmetry of the domain $\mathbb{R}^m$ across the hyperplane $x^1 = 0$ and that of the codomain $\mathbb{R}$ across the origin; hence the appropriate version for harmonic maps is:

<u>Theorem 5.3.5 (First Reflection Principle for Harmonic Maps)</u>

Let M, N be smooth Riemannian manifolds with isometric symmetries r, s across hypersurfaces $H \subset M$, $K \subset N$ respectively. Let U be a domain adjacent to H which is disjoint from its "reflection" r(U), and write $B = \operatorname{int}(\overline{U} \cap H) = \operatorname{int}(\overline{r(U)} \cap H)$ for the common boundary of U and r(U).

Let $f: U \to N$ be a harmonic map such that:

(1) f has a C^2 extension to $U \cup B$, which we shall also denote by f;

(2) $f(B) \subset K$;

(3) the derivative $df(p)$ maps normals to H to normals to K, $\forall p \in B$;

then we can extend f to a harmonic map on $U \cup r(U) \cup B$ by the formula:

$$f(r(p)) = s(f(p)) \qquad (p \in U \cup B).$$

<u>Proof</u>

Since r, s are isometries, the formula certainly defines a harmonic map $\tilde{f}$ on r(U). We wish to show that we can glue f and $\tilde{f}$ together using Lemma (5.3.2). But conditions (1) and (2) of (5.3.2) are satisfied by hypothesis, and since $\tilde{f} = s \circ f \circ r$, hypothesis (3) above implies:

$$\frac{\partial f}{\partial n}(p) = \frac{\partial \tilde{f}}{\partial n}(p) \quad \forall\, p \in H.$$

Thus condition (3) of (5.3.2) is satisfied and the present result follows from Lemma (5.3.2). ∎

Now a second classical reflection principle asserts that if f is a harmonic function, defined as before on a domain U adjacent to the hyperplane H: $x^1 = 0$ in $\mathbb{R}^m$, admitting a continuous extension to $U \cup B$ where $B = \operatorname{int}(\bar{U} \cup H)$, and such that the normal derivative to H vanishes on H, then f has an extension across the hyperplane by the reflection law:

$$f(x^1, x^2, \ldots, x^m) = f(-x^1, x^2, \ldots, x^m)$$

Note that, in contrast to the first classical reflection principle discussed above, we are only using the symmetry of the domain $\mathbb{R}^m$; thus the appropriate generalisation to harmonic maps is:

<u>Theorem 5.3.6 (Second Reflection Principle for Harmonic Maps)</u>

Let M be a smooth Riemannian manifold with isometric symmetry $r: M \to M$ across a hypersurface H. Let U be a domain adjacent to H which is disjoint from its "reflection" $r(U)$. Write $B = \operatorname{int}(\bar{U} \cup H) = \operatorname{int}(\overline{r(U)} \cup H)$ for the common boundary of U and $r(U)$.

Let $f: U \to N$ be a harmonic map such that:

(1) f has a C^2 extension to $U \cup B$, which we shall also denote by f;

(2) $df(p)$ maps normals to H to zero $\forall\, p \in B$;

then f can be extended to a harmonic map on $U \cup r(U) \cup B$ by the formula:

$$f(p) = f(r(p)) \quad (p \in U \cup B).$$

<u>Proof</u>

Since r is an isometry, the formula defines a harmonic map $\tilde{f}$ on $r(U)$. Again we wish to glue f and $\tilde{f}$ together using Lemma (5.3.2). Conditions (1) and (2) of Lemma (5.3.2) are clearly satisfied; further, by hypothesis (2) above:

$$\frac{\partial f}{\partial n}(p) = \frac{\partial \tilde{f}}{\partial n}(p) \quad (= 0) \quad \forall p \in B,$$

therefore condition (3) of Lemma (5.3.2) is also satisfied and the present result follows from that lemma. ∎

An example of a harmonic map exhibiting this sort of symmetry is the map $\underline{\mathbb{R}}^2 \to \underline{\mathbb{R}}^2$, $(x,y) \mapsto (x^2 - y^2, y)$; also Smith's map [Sm p.95] of the torus to the sphere where H = meridian of torus, f(H) = closed curve parallel to a great circle.

D Conformal Reflection and Doubling

Let W be a bordered Riemann surface [A-S]. In this section we shall show how we can sometimes extend a harmonic map given on W to the "double" of W. We first describe briefly the process of doubling W; for a fuller treatment see Ahlfors and Sario [A-S pp.117 - 9].

The conjugate Riemann surface $\tilde{W}$ (†) for the bordered Riemann surface W is defined as follows. The underlying set of points of $\tilde{W}$ is the same as that of W. Let $r: W \to \tilde{W}$ be the identity map. Then for each chart

$$h: U \to U' \subset \underline{\mathbb{C}}$$

for W form the chart

$$-\tilde{h}: \tilde{U} \xrightarrow{r^{-1}} U \xrightarrow{h} U' \to -\bar{U}' \subset \underline{\mathbb{C}}$$
$$z \mapsto -\bar{z}$$

Here $\tilde{U} = r(U)$, $-\bar{U}' = \{-\bar{z}: z \in U'\}$, and $\bar{z}$ denotes the complex conjugate of z. W has thus been given the same topological structure but the "opposite" complex structure; in fact the identity map $r: W \to \tilde{W}$ is conformal and orientation reversing (*).

We now weld W and $\tilde{W}$ together by identifying corresponding points on their borders; this gives a topological surface $W \cup \tilde{W}$ (without boundary). We give $W \cup \tilde{W}$ a complex structure as follows: at a point p in the interior of W or $\tilde{W}$, take a chart about p as a point of W or $\tilde{W}$; this gives a chart for p as a point of $W \cup \tilde{W}$; at a point $p \in \partial W = \partial\tilde{W}$ on the common border, first choose a chart about p as a point of W:

$$h: U \to U' \subset \underline{\mathbb{C}}$$

where U' is a relatively open subset of the upper half-plane of $\underline{\mathbb{C}}$. Note that h takes points of ∂W to the real axis of $\underline{\mathbb{C}}$. As explained above, this gives a chart about p as a point of $\tilde{W}$:

$$-\tilde{h}: \tilde{U} \to -U' \subset \underline{\mathbb{C}}$$

It is more convenient to modify this chart by composing with the map

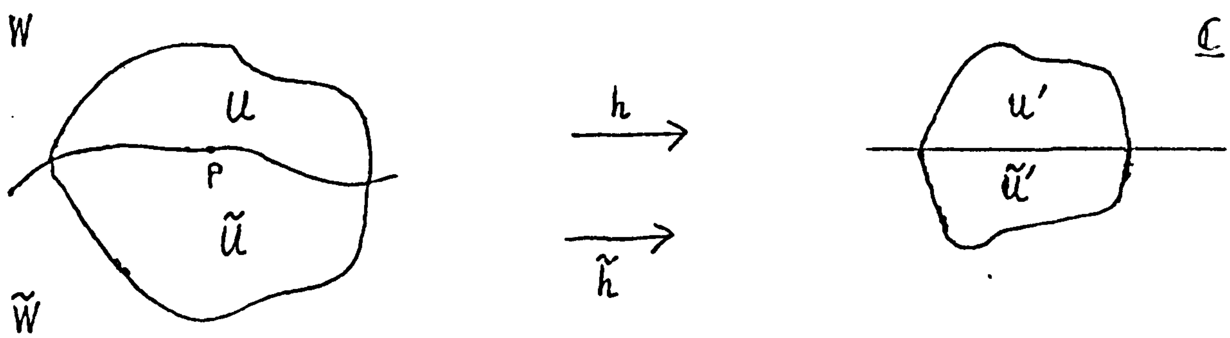
(†) We use the notation $\tilde{W}$ in preference to $\bar{W}$ used by Ahlfors and Sario to avoid confusion with "closure".

(*) and thus r is anti-holomorphic.

$-\tilde{U}' \to \tilde{U}'$ $z \mapsto -z$ (where $\tilde{U}' = \{\bar{z}: z \in U'\}$) to give:

$$\tilde{h}: \tilde{U} \to \tilde{U}' \subset \mathbb{C}.$$

Now h and $\tilde{h}$ agree on ∂W and can therefore be glued together (cf. (5.3.2)) to give a chart about p as a point of $W \cup \tilde{W}$ (see diagram below).



It is readily seen that the charts we have defined give $W \cup \tilde{W}$ a complex structure compatible with those of W and $\tilde{W}$; $W \cup \tilde{W}$ is thus a Riemann surface (without border) which is called the <u>double</u> of W.

Examples: the double of a closed disc is a sphere; the double of an annulus $S^1 \times I$ is a torus.

We now investigate when a harmonic map defined on a bordered Riemann surface can be extended to a map on its double.

<u>Theorem 5.3.7 (Conformal Reflection Principle)</u>

Let W be a bordered Riemann surface and let N be a smooth Riemannian manifold with isometric symmetry $s: N \to N$ about a hypersurface K. Let $f: \text{int } W \to N$ be a harmonic map such that:

(1) f has a C^2 extension to W ($= \text{int } W \cup \partial W$), which we shall also denote by f

(2) $f(\partial W) \subset K$;

(3) $df(p)$ maps normals to normals for all $p \in B$;

then we can extend f to a harmonic map on the double $W \cup \tilde{W}$ by the formula:

$$f(r(p)) = s(f(p)) \qquad (p \in W)$$

where r is the identity map $r: W \to \tilde{W}$.

<u>Proof</u>

Since r is anti-holomorphic and s is an isometry, then by (1.3.9)

Cont.d on p156

E A Possible Application of the Reflection Principle

In this section, we shall indicate how the reflection principles might be used to construct harmonic maps between surfaces exhibiting conformal or isometric symmetry if we can find harmonic maps between certain planar regions satisfying certain conditions. As an example, we shall show how, given a suitable harmonic map from a square to a hemisphere, we can use the reflection laws to get a map from the torus to the sphere.

The method requires us to know a solution to the following boundary-value-type problem:

Problem 5.3.8

Let Q be the unit square in $\mathbb{R}^2$, and let E be the equator of the 2-sphere S^2. Find a harmonic map $f:(Q,\partial Q) \to (S^2,E)$ such that:

(1) f is C^2 up to the boundary ∂Q of Q;

(2) df(p) takes normals to ∂Q to normals to E. (*)

Notes

(1) It suffices to find such a map $f:(D^2,S^1) \to (S^2,E)$ from the unit disc D^2, for we can then compose with a holomorphic map $(Q,\partial Q) \to (D^2,S^1)$.

(2) It suffices to find a map into the northern hemisphere, in which case we can formulate the problem as finding maps $f:(D^2,S^1) \to (D^2,S^1)$ satisfying the appropriate tension field equations (we are considering the northern hemisphere as being a disc with a certain metric). Put this way, the problem has similarities with one discussed by Heinz [He 2], [He 3].

(3) One solution of the problem is the Weierstrass $\wp$ function $f:(Q,\partial Q) \to (S^2,E)$ given by [Bi]:

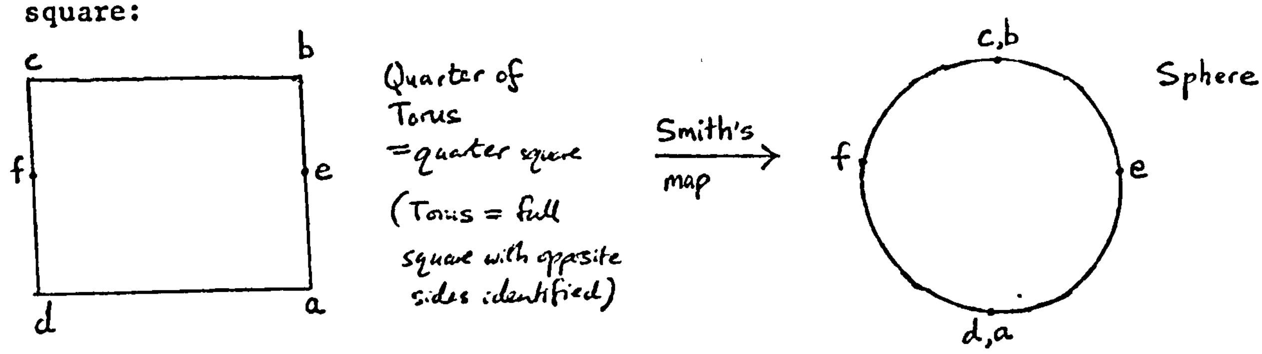
$$f(z) = \wp(z;2,2) = \frac{1}{z^2} + \sum_{s,t}\left\{\frac{1}{(z-s-t)^2} - \frac{1}{(s+t)^2}\right\}$$

Here $z = x + iy$, $(x,y)\in Q\subset\mathbb{R}^2$, and S^2 is identified with the extended

(*) At the corners, df must take the normals to both sides to normals to E complex plane in the usual way.

This map also provides a holomorphic map for note (1) above.

(4) A solution to the problem which is not holomorphic is provided by Smith's map of the torus to the sphere [Sm p.95] restricted to a quarter square:

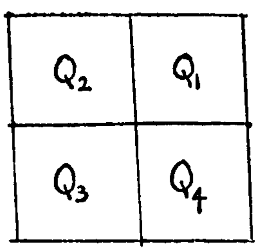


Here two opposite sides of the square collapse to the poles of the sphere. Of course, in this case we have already got a map from the torus to the sphere, but it does demonstrate that there exists a non-holomorphic solution to our problem. ∎

Now let f be a solution to our problem. Divide the unit square Q into four quadrants Q_1, Q_2, Q_3, Q_4 as shown below. It will be more convenient to re-define our given harmonic map as a map on the first quadrant, viz:

$$f:(Q_1,\partial Q_1) \to (S^2,E)$$

Q:



Proposition 5.3.9

f can be extended by reflection to the whole of $\mathbb{R}^2$ to give a map $\mathbb{R}^2 \to S^2$ such that $f(x+r,y+s) = f(x,y)\ \forall r,s \in \mathbb{Z}$.

Proof

We simply use the isometric symmetry of $\mathbb{R}^2$ across the sides of Q_1, the isometric symmetry of S^2 across E, and Theorem (5.3.5) to extend f to $Q_1 \cup Q_2$ and thence to Q. Clearly by repeating this procedure, we get a harmonic map on the whole of $\mathbb{R}^2$, and the formula for reflection in (5.3.5) leads to the claimed periodicity. ∎

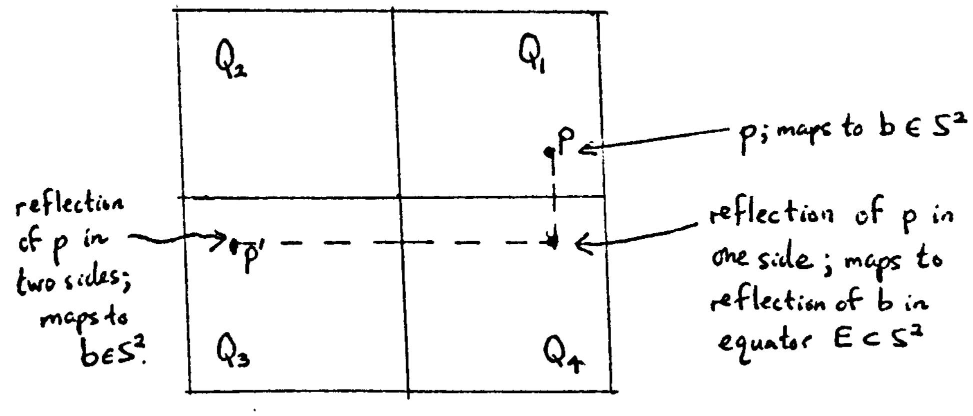
Thus f gives rise to a harmonic map $\hat{f}: S^1 \times S^1 \to S^2$ of the torus to the sphere.

Proposition 5.3.10

The degree of $\hat{f}: S^1 \times S^1 \to S^2$ is even (it may be zero).

Proof

Let $b \in S^2$ be a regular value of $\hat{f}$ with $b \in \hat{f}(\text{int } Q_1)$. Then $\hat{f}^{-1}(b)$ contains a point $p \in \text{int } Q_1$. By the way $\hat{f}$ is constructed, it also contains a point $p' \in \text{int } Q_3$ such that p' is the reflection of p in a side of Q_1 and a side of Q_4 as shown. Note p' and p are distinct points. Thus $\hat{f}^{-1}(b)$ consists of an even number of points. It is now clear from the definition (§3.2B) that the degree of $\hat{f}$ must be even (†).



∎

Notes

(1) Solving problem (5.3.8) for a rectangle $[0,\frac{1}{2}a]\times[0,\frac{1}{2}b]$ instead of a square, and applying (5.3.9) would give a harmonic map $f: \mathbb{R}^2 \to S^2$ with $f(x+ra, y+sb) = f(x,y)\ \forall\, r,s \in \mathbb{Z}$. Factoring $\mathbb{R}^2$ by the subgroup $\{(ra,sb): r,s \in \mathbb{Z}\}$ would give a harmonic map on a different torus. Further we can factor $\mathbb{R}^2$ instead by the subgroup generated by (pa,qb), (p'a,q'b) where p, q, p', q' are any fixed integers; then by suitable

(†) In fact we can show, with the obvious extension of the definitions, that $\deg(\hat{f}) = 2\deg(f)$.

choice of a, b, p, q, p', q' we can get a harmonic map from any standard torus: $\mathbb{R}^2$/discrete subgroup to the sphere. Since an arbitrary torus (= arbitrary compact Riemann surface of genus 1) is conformally equivalent to a standard torus $\mathbb{R}^2$/discrete subgroup, we have constructed harmonic maps from an arbitrary torus to the sphere. Composing with holomorphic maps from higher genus surfaces into the torus then gives harmonic maps from higher genus Riemann surfaces into the sphere.

(2) In a similar fashion we might construct harmonic maps from the torus to the sphere by finding suitable harmonic maps from an annulus to a hemisphere and using the conformal reflection principle. Such a map would not necessarily have even degree, and might provide a way of constructing harmonic maps from the torus to the sphere of degree 1. $\square$

Thus the solution of problem (5.3.8) and similar "planar" boundary-value-type problems is seen to be of great importance and should be worthy of further study.

cont'd from p151.

the formula defines a harmonic map $\tilde{f}$ on int $\tilde{W}$. Let $f \cup \tilde{f}: W \cup \tilde{W} \to N$ denote the map which extends f and $\tilde{f}$; $f \cup \tilde{f}$ is C^0 on $W \cup \tilde{W}$ and harmonic on $W \cup \tilde{W} \setminus \partial W$. Let $p \in \partial W$. We show that $f \cup \tilde{f}$ is harmonic in a neighbourhood of p. Choose a chart

$$h \cup \tilde{h}: U \cup \tilde{U} \to U' \cup \tilde{U}' \subset \underline{\mathbf{C}}$$

about p as above. Pull back the metric of $\underline{\mathbf{C}}$ using $h \cup \tilde{h}$ to give a smooth Riemannian metric for $U \cup \tilde{U}$. Note this makes $r|_U$ an isometry. The hypotheses above now allow us to apply the first reflection principle (5.3.5) to show that $f \cup \tilde{f}$ is harmonic on $U \cup \tilde{U}$. Thus $f \cup \tilde{f}$ is harmonic on a neighbourhood of $p \in \partial W$, and since p is arbitrary, $f \cup \tilde{f}$ is harmonic on $W \cup \tilde{W}$. $\square$

APPENDICES AND

LIST OF REFERENCES

Appendix 1 CANONICAL FORM FOR A COLLAPSE POINT

We here prove:

Proposition 2.1.18

If p is a collapse point for the smooth mapping $f:M\to N$ between smooth surfaces M,N, then there exist smooth coordinates (x,y) centred on p, (u,v) centred on $f(p)$ such that f assumes the form:

$$u = xy \ , \ v = y \ .$$

Proof

By definition of a collapse point (see (2.1.15)), p lies on a general fold σ such that, for some neighbourhood U of p, $f(U\cap\sigma)$ = point. Now, since for a good singular point $p\in M$, $df(p)$ has rank 1 (by (2.1.4)), $\ker df(p)$, $\operatorname{im} df(p)$ define unique directions in their respective tangent spaces. Thus we may choose smooth coordinates (x,y), (u,v) centred on p, $f(p)$ such that:

(1) the x-axis is in the direction $\ker df(p)$,

(2) the v-axis is in the direction $\operatorname{im} df(p)$ and $\frac{\partial v}{\partial y}(p) = 1$.

Such coordinates are called Whitney special coordinates (see [W]) and are characterised by having $df(p) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Following Whitney we say that a smooth function $R(x,y)$ has $\operatorname{ord}(R)\geqslant r$ iff all its partial derivatives of order $< r$ vanish at $(0,0)$ (†). In Whitney special coordinates, f has the form:

$$u = R(x,y)\ , \quad (\operatorname{ord} R\geqslant 2)\ , \ v=y + S(x,y)\ , \ (\operatorname{ord} S\geqslant 3)$$

We shall now simplify this form by carrying out smooth changes of coordinates. First, set:

$$x' = x \ , \ y' = y + S(x,y) \ ;$$

then [W14.17] :

$$x = x' \ , \ y = y' + S(x',y') \ , \ (\operatorname{ord} S'\geqslant 3)$$

Hence, dropping primes, f has the form:

$$u = R(x,y) \ , (\operatorname{ord} R\geqslant 2) \ , \ v = y \ .$$

(†) In the notation of (1.4.2), we see that $\operatorname{ord}(R)\geqslant r \Leftrightarrow R=o(\rho)^{r-1} \Leftrightarrow R=O(\rho)^{r}$.

Now, since $u(x,y) = 0$, $v(x,y) = 0$ for $(x,y) \in \sigma \cap U$, it is evident that $\sigma \cap U$ must be given by $y=0$ and further[W,14.8] :

$$R(x,y) = yW(x,y) \qquad (\text{ord } W \geqslant 1)$$

Thus:

(6.1A) $\quad u = yW(x,y)$, (ord $W \geqslant 1$) , $v = y$.

Now a trivial calculation shows:

$$dJ(0,0) = (J_x , J_y)(0,0) = (u_{xx}(0,0) , u_{xy}(0,0))$$

where J_x denotes $\partial J/\partial x$ etc. Since p is a good singular point, $dJ(0,0) \neq 0$. But from (6.1A) $u_{xx}(0,0) = 0$ and therefore $u_{xy}(0,0) \neq 0$, i.e. $W_x(0,0) \neq 0$. Thus:

$$x' = W(x,y) , \; y' = y$$

defines a non-singular smooth change of coordinates, in which, after dropping primes, f has the form:

$$u = xy , \; v = y$$

as desired.

WHITNEY SPECIAL COORDINATES

Appendix 2 A HEURISTIC DESCRIPTION OF A SPECIFIC HARMONIC MAP

This section uses the work of section 5.2 to construct a harmonic map from a Riemann surface M_2 (compact, of genus 2) onto the torus $T=S^1\times S^1$ which exhibits folding along a single smooth general fold. We claim no great rigour for this section, which is intended to give a picture of a specific harmonic map.

Let M_2 be given as a surface in $\underline{R}^3$ as in fig. (6.2A) with two orthogonal planes of symmetry - the xz plane (represented by the page) and the xy plane (orthogonal to the page). The Weierstrass points $W_1, W_2, \dots, W_6$ are at the intersections of these planes (for a Weierstrass point is fixed under any conformal involution and two such involutions are reflection across the xz and xy planes). We can also think of M_2 as the Riemann surface of the function $k^2 = (z-e_2)\dots(z-e_5)$ [Sp] where $e_2, \dots, e_5 \in \underline{\mathbb{C}}$ are real. Then M_2 is a ramified covering $k: M_2 \to S$ of the Riemann sphere of degree 2 with branch points of order 1 at the Weierstrass points $W_1, \dots, W_6$. These points have images $e_1=\infty, e_2, \dots, e_5 \in \underline{\mathbb{C}}\cup\infty$. (Here, as usual, we identify the Riemann sphere with the extended complex plane $\underline{\mathbb{C}}\cup\infty$.)

Let a_1, b_1, a_2, b_2 denote the generators of the fundamental group $\pi_1(M_2)$ and let c_1, d_1 denote the generators of $\pi_1(T)$.
We wish to describe the harmonic map $f: M_2 \to T$ in the homotopy class which maps the generators of the fundamental group as follows:

$$a_1 \mapsto c_1 \ , \ b_1 \mapsto d_1 \ , \ a_2 \mapsto 0, \quad b_2 \mapsto 0$$

Here 0 denotes the identity element of $\pi_1(T)$. Thinking of M_2 as the torus with an attached handle (the one spanned by a_2 and b_2), this map f must "squash" the handle onto the torus. We shall show that this is accomplished by "folding" along a single closed general fold (see fig. 6.2A). Note that f exists by (1.i.ii) and is unique up to translations of T [Fu]
Also note that f is real-analytic (by (1.3.8)) and of degree 1 (evident).

We construct f as in §5.2 by choosing a pair of harmonic

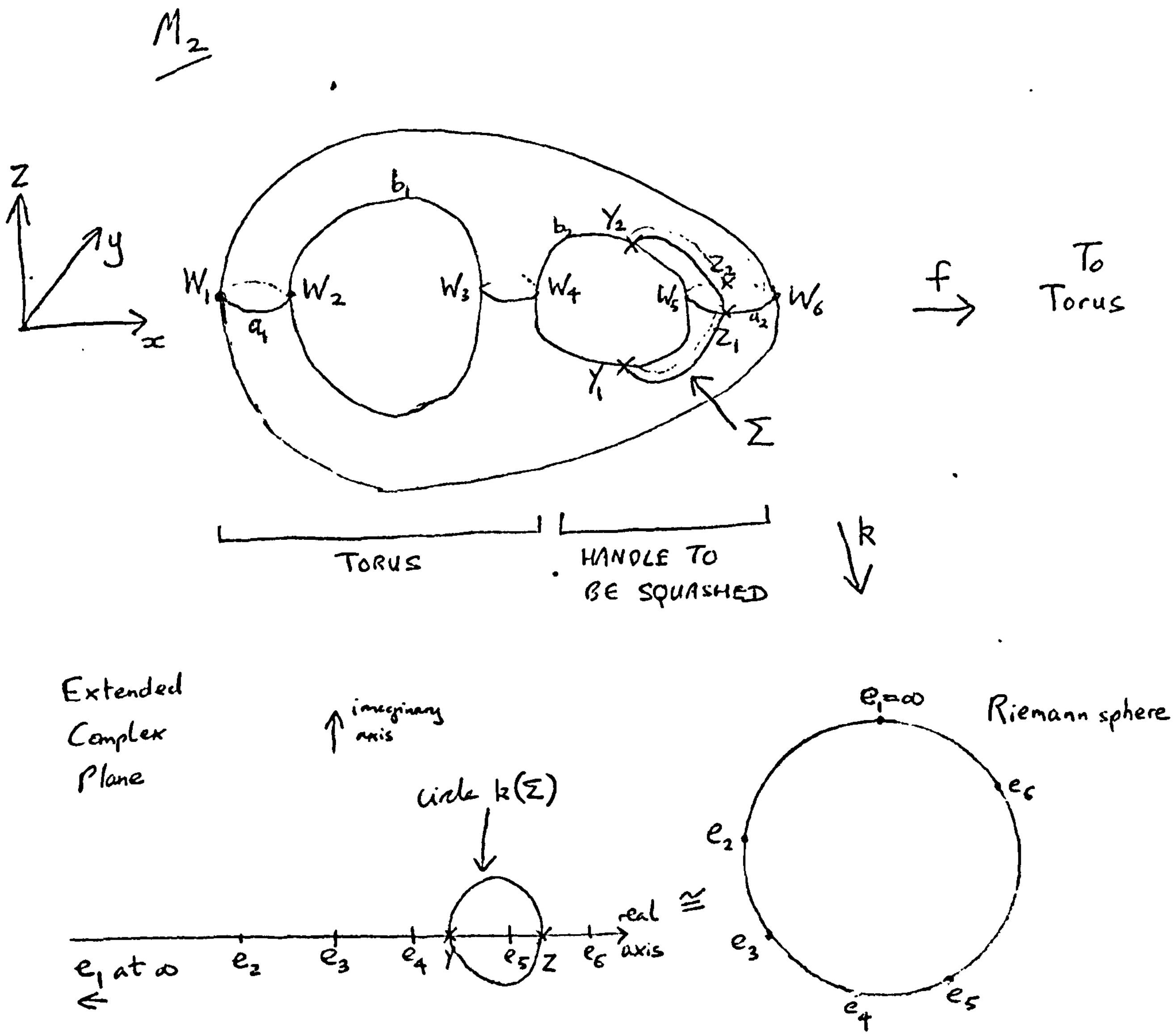


FIG 6.2A

WEIERSTRASS POINTS AND THEIR IMAGES UNDER k. ALSO SINGULAR SET Σ FOR f AND ITS IMAGE UNDER k

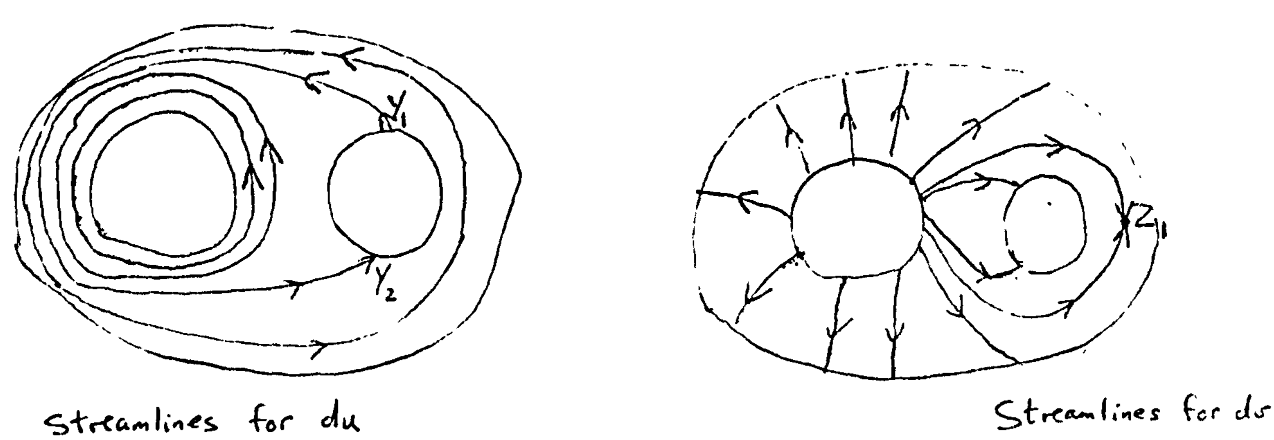


FIG 6.2B

STREAMLINES FOR THE HARMONIC DIFFERENTIALS du, dv ON M_2

differentials on M_2. By the theory in §5.2 the harmonic differentials, which we denote by du, dv have periods as follows:

du has period 1 around a_1 , 0 around b_1, a_2, b_2

dv has period 1 around b_1 , 0 around a_1, a_2, b_2

The streamlines for u and v (= the lines which are the integral curves of grad u and grad v - see [Sp]) are shown in fig. (6.2B) taken from Springer [Sp p.27]. The essential point is that du has zeros at the symmetrically placed points Y_1, Y_2 which lie over the point $k(Y_1) = k(Y_2) = Y$ lying on the real axis of $\underline{C} \cup \infty = S$ between e_4 and e_5 , and dv has zeros at the symmetrically placed points $Z_1, Z_2 \in M_2$ which lie over the point $k(Z_1) = k(Z_2) = Z$ on the real axis of $\underline{C} \cup \infty = S$ between W_5 and W_6 . We now show:

(6.2C) Σ is the inverse image under $k: M_2 \to \underline{C} \cup \infty = S$ of a circle through Y and Z.

For let $\theta = du + i{*}du$, $\phi = dv + i{*}dv$ (see §5.2). Then since θ and ϕ have disjoint zero sets, they must be linearly independent (trivial) and thus we have case 4 of theorem (5.2.14). Thus if θ/ϕ is the meromorphic function on M_2 as defined in §5.2, and $\widehat{\theta/\phi}$ denotes the associated holomorphic map $M_2 \to S$, the singular set Σ of $f: M_2 \to T$ is given by:

$$\Sigma = (\widehat{\theta/\phi})^{-1}(\text{real circle of } S) \cup \text{branch points of } f$$

Since f has degree 1, it can have no branch points, and therefore

$$\Sigma = (\widehat{\theta/\phi})^{-1}(\text{real circle of } S).$$

We now calculate θ/ϕ.

Now we may express θ and ϕ in terms of a basis for the holomorphic differentials [Sp p.293] viz:

$$\theta = a\,dz/k + bz\,dz/k \quad ; \quad \phi = c\,dz/k + dz\,dz/k$$

where $a,b,c,d \in \underline{C}$, $ad-bc \neq 0$. Then

$$\theta/\phi = \frac{az+b}{cz+d}$$

showing that θ/ϕ = k followed by a bilinear transformation of $\underline{C} \cup \infty$. Now it is well-known that such a transformation transforms lines and circles in the extended complex plane into lines and circles, hence:

$$\Sigma = (\widehat{\theta/\rho})^{-1}(\text{real circle of } S) = k^{-1}(\text{some circle or line of } \underline{C}\cup\infty)$$

Now, the zeros of du and dv are singularities of f, therefore the circle or line of $\underline{C}\cup\infty$ passes through Y and Z. That it should be a <u>line</u> through Y and Z is clearly impossible, for this would give f the wrong homotopy class, thus it is a <u>circle</u> through Y and Z.

The circle is shown in fig.(6.2A) . It is, of course, symmetrical about the real axis. It is seen to encircle precisely one branch value of $k:M_2 \to \underline{C}\cup\infty$. Therefore it is clear that Σ consists of just one C^{ω} closed 1-submanifold passing through Y_1, Z_2, Y_2, Z_1. By (3.1.7) Σ is a general fold. We now show:

(6.2D) Σ contains no collapse points

For if Σ had any collapse points, then by analytic continuation the whole of Σ would be a collapse line. Consider the image of the disc D enclosed by Σ (see fig (6.2)). Its boundary maps under f to a point , but its interior maps without singularities. Factoring by ∂D we thus get a local homeomorphism of a sphere $D/\partial D$ into the torus. This is impossible.

Thus f has no collapse points. There may of course be good singular points of order r, $2 \leq r < \infty$, by (2.1.14) these are isolated; thus:

(6.2E) Σ consists of fold points except for a finite number of good singular points of order r ($2 \leq r < \infty$).

Thus our harmonic map $f:M_2 \to T$ exhibits folding along a single closed general fold Σ described by (6.2E).

This completes our description of the harmonic mapping f. ■

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